Math 252: Representation Theory

Exercises XVI

Problem 5. *Let G be a finite non-abelian simple group. Show that any subgroup $H \subseteq G$ of prime-power index is centerless (that is, $Z(H) = \{1\}$). Deduce from this that G has no abelian subgroup of prime-power index.*

Solution. For contradiction, assume that $Z(H) \neq \{1\}$. Choose any $1 \neq h \in Z(H)$. Since h is central in H, we know that $H \subseteq C_G(h)$, and therefore $|C_G(h)| = [C_G(h) : H]\,|H|$.

Recall the size of the conjugacy class of h is

$$|h| = \frac{|G|}{|C_G(h)|} = \frac{|G|}{[C_G(h) : H]\,|H|} = \frac{[G : H]}{[C_G(h) : H]}.$$

However $[G : H]$ is a prime power, and thus $|h| = p^e$ for some prime p. Now it cannot be that $|h| = 1$, as then $h \in Z(G)$ which is trivial by hypothesis. Therefore $|h| = p^e$ for some $e \geq 1$, and by a theorem of Burnside we conclude that G is not simple, a contradiction.

We easily deduce that G contains no abelian subgroups of prime-power index, as if $H \subseteq G$ is abelian and has prime-power index then $H = Z(H) = \{1\}$ and thus G is a p-group. In particular, we note $Z(G) \neq \{1\}$, a contradiction.

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