Math 252: Representation Theory

Exercises XVI

Problem 2 (Burnside)**.** *Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1) > 1$. Show that $\chi(g) = 0$ for some $g \in G$. (Hint: Deny the conclusion and apply the Arithmetic-Geometric inequality of the $|G| - 1$ numbers $|\chi(g)|^2$ $(g \neq 1)$.)*

Solution. Observe

$$0 \leq \prod_{g \neq 1} |\chi(g)|^2 \leq \left(\frac{\sum_{g \neq 1} |\chi(g)|^2}{|G| - 1} \right)^{|G|-1} = \left(\frac{|G| - |\chi(1)|^2}{|G| - 1} \right)^{|G|-1} < 1^{|G|-1} = 1$$

where the second inequality arises from the Arithmetic-Geometric inequality, and the middle equality arises by noting that $\sum_g |\chi(g)|^2 = |G|$ since $\chi \in \mathrm{Irr}(G)$.

In Problem 1 we showed that $\prod_{g \neq 1} |\chi(g)|^2 \in \mathbb{Z}$, and thus $\prod_{g \neq 1} |\chi(g)|^2 = 0$ since it is non-negative and strictly less than 1.

Since the product is zero, we conclude that $\chi(g) = 0$ for some $g \in G$.

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