Math 252: Representation Theory

Exercises XV

Problem 5. *Show that the Feit-Thompson Theorem ("Odd Groups are Solvable") is equivalent to the statement that finite non-abelian simple groups have even order.*

Solution.

"$\Rightarrow$": Let G be a non-abelian simple group. Suppose for contradiction that G has odd order. Then G has a normal series whose factor groups are abelian. In particular, either G has a non-trivial proper normal subgroup or G is itself abelian. Either case is a contradiction.

"$\Leftarrow$": Let G be a group of odd order. We use total induction on $|G|$ to show that G is solvable. Clearly the trivial group is solvable. Suppose that all groups of odd order strictly less than $|G|$ are solvable. Since $|G|$ is odd, G cannot be a non-abelian simple group. Therefore either:

- G is abelian, and hence solvable; or
- G has a non-trivial proper normal subgroup N. But N and G/N both have odd order less than $|G|$ and hence are solvable by induction, and therefore G is solvable.

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