

## Math 252: Representation Theory

### Exercises XV

**Problem 3.** Let  $U, V, W$  be simple  $kG$ -modules where  $k$  is a splitting field of characteristic 0 for  $G$ . Let  $T$  be the regular module  $kG$ .

- (a) Show that  $U \otimes T \cong (\dim U)T$
- (b) Show that  $U \otimes V$  can be embedded in  $T$ .
- (c) Show that  $W$  appears in  $U \otimes V$  at most  $\dim W$  times.

### Solution.

- (a) Recall that  $\chi_{U \otimes T} = \chi_U \chi_T$ . Since  $\chi_T = (|G|, 0, 0, \dots)$ , we conclude  $\chi_U \chi_T = (\chi_U(1))\chi_T = (\dim U)\chi_T$ , and hence  $U \otimes T \cong (\dim U)T$  since their characters agree.

Remark: This proof did not require that  $U$  is a simple  $kG$ -module.

- (b) It suffices to show that for all  $\varphi \in \text{Irr}(G)$  we have  $\langle \chi_{U \otimes V}, \varphi \rangle \leq \langle \chi_T, \varphi \rangle$ . We show this by explicit calculation (using the Cauchy-Schwarz inequality and noting that  $U$  and  $V$  are absolutely irreducible):

$$\begin{aligned} \langle \chi_{U \otimes V}, \varphi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_U(g) \chi_V(g) \bar{\varphi}(g) \\ &\leq \frac{1}{|G|} \sum_{g \in G} |\chi_U(g)| |\chi_V(g)| |\varphi(g)| \\ &\leq \frac{\varphi(1)}{|G|} \sum_{g \in G} |\chi_U(g)| |\chi_V(g)| \\ &\leq \frac{\varphi(1)}{|G|} \left( \sum_{g \in G} |\chi_U(g)|^2 \right)^{1/2} \left( \sum_{g \in G} |\chi_V(g)|^2 \right)^{1/2} \\ &\leq \varphi(1) = \langle \chi_T, \varphi \rangle. \end{aligned}$$

- (c) We know that  $W$  appears in  $T$  exactly  $\dim W$  times. Since  $U \otimes V$  may be embedded in  $T$ , we conclude that  $W$  appears in  $U \otimes V$  at most  $\dim W$  times.

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Solution by Bradley Froehle (bfroehle@math.berkeley.edu).