Math 252: Representation Theory

Exercises XV

Problem 1.

(a) Let $\varphi = (2, 0, -1) \in \mathrm{Irr}(G)$ where $G = S_3$. Show that the character ring $\mathrm{Ch}(G)$ (consisting of all virtual characters) is equal to $\mathbb{Z}[\varphi]$ (the ring generated by φ).

(b) Let $\varphi = (3, -1, 0, 0) \in \mathrm{Irr}(G)$ where $G = A_4$. Show that $\mathrm{Ch}(G)$ is not equal to $\mathbb{Z}[\varphi]$.

Solution.

(a) Let $G = S_3$. Recall that $\mathrm{Irr}(G) = \{1_G, \chi, \varphi\}$ where $\chi = (1, -1, 1)$. Observe that $1_G, \varphi \in \mathbb{Z}[\varphi]$. One can easily verify that $\chi = \varphi^2 - \varphi - 1_G$, and hence $\chi \in \mathbb{Z}[\varphi]$.

Since $\mathbb{Z}[\varphi]$ contains all elements of $\mathrm{Irr}(G)$, we conclude that $\mathrm{Ch}(G) = \mathbb{Z}[\varphi]$.

(b) Let $G = A_4$. Recall that $\mathrm{Irr}(G) = \{1_G, \chi, \bar{\chi}, \varphi\}$ where $\chi = (1, 1, \omega, \omega^2)$. However $\mathbb{Z}[\varphi]$ contains only *real* virtual characters, and hence $\chi, \bar{\chi} \notin \mathbb{Z}[\varphi]$. Thus $\mathrm{Ch}(G) \neq \mathbb{Z}[\varphi]$.

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