Exercise XV.4
(Frobenius Fan)

Ex. XV.4. *Prove the following generalization of Frobenius' Integrality Theorem: For* $\chi \in \mathrm{Irr}(G)$, $g \in G$, *and* $d = \gcd(\chi(1), |g|)$, $d \cdot \chi(g)/\chi(1)$ *is an algebraic integer.*

Solution. Write $|g| = ad$, $\chi(1) = bd$ $(a, b \in \mathbb{N})$, and say $1 = ma + nb$ $(m, n \in \mathbb{Z})$. Multiplying this equation by $\alpha := \chi(g)/b$, we get

$$\alpha = m\,\frac{a}{b}\,\chi(g) + n\,\chi(g) = m \cdot \frac{|g|}{\chi(1)}\,\chi(g) + n\,\chi(g),$$

which is an algebraic integer (by FIT). This does it, as $\alpha = d\,\chi(g)/(bd) = d \cdot \chi(g)/\chi(1)$.