MATH 252 ASSIGNMENT 17 PROBLEM 8

ALEX FINK

Problem. Let $H \lhd G$ and $\mu \in \mathrm{Irr}(G)$. Show that $\mu = \nu^G$ for some $\nu \in \mathrm{Irr}(H)$ iff $\mu(G \setminus H) = \{0\}$ and μ_H is a sum of distinct characters in $\mathrm{Irr}(H)$.

Solution.
To ($\Leftarrow$). Let $\mu \in \mathrm{Irr}(G)$ be a character zero away from H such that μ_H is a sum of distinct characters. The first condition gives us that for any character χ on G we have $[\mu, \chi]_G = [G : H]^{-1}[\mu_H, \chi_H]_H$, since all terms in the sum defining $[\mu, \chi]_G$ for elements $g \notin H$ are zero and can be dropped. In particular $[\mu_H, \mu_H] = [G : H][\mu, \mu] = [G : H]$. Let $\nu_1, \ldots, \nu_r \in \mathrm{Irr}(H)$ be the distinct characters such that $\mu_H = \sum_{i=1}^r \nu_i$. Then $[\mu_H, \mu_H] = r$ by orthonormality of the basis of irreducibles, so that $r = [G : H]$.

Let $\nu \in \mathrm{Irr}(G)$ be the one of the ν_i such that $\nu(1)$ is minimal; in particular

$$\nu(1) \le \frac{1}{r}(\sum_i \nu_i(1)) = \frac{1}{r}\mu_H(1) = \frac{1}{r}\mu(1).$$

Frobenius reciprocity gives $1 = [\mu_H, \nu]_H = [\mu, \nu^G]$. That is, the irreducible μ occurs once in ν^G. But $\nu^G(1) = [G : H]\nu(1) = r\nu(1) \le \mu(1)$. Now ν^G cannot possibly have μ as a component if $\nu^G(1) < \mu(1)$. Therefore $\nu^G(1) = \mu(1)$, and even this requires that μ be the only irreducible component of ν^G, more plainly $\mu = \nu^G$. This is what was to be shown.

To ($\Rightarrow$). Let $\nu \in \mathrm{Irr}(H)$ be a character such that $\mu = \nu^G$ remains irreducible. We know that $\mu(G \setminus H) = 0$ by our formula for computing induced characters: by normality of H, elements of $G \setminus H$ are outside all conjugates of H.

Now $[\mu_H, \mu_H] = [G : H][\mu, \mu] = [G : H]$, the left equality holding for the same reason as before and the right because μ is (absolutely) irreducible. On the other hand, by construction of the induced character, $\mu_H = \sum_{s \in S}(g \mapsto \nu(sgs^{-1})$ where S is a set of representatives for G/H. Each of these characters $\nu^s := g \mapsto \nu(sgs^{-1})$ is irreducible, so this expresses μ_H as a sum of $[G : H]$ irreducibles. We conclude that since $[\mu_H, \mu_H] = [G : H]$, all the cross-terms in the expansion of $[\mu_H, \mu_H] = \sum_{s,t \in S}[\nu^s, \nu^t]$ vanish, so that $\nu^s \ne \nu^t$ for $s \ne t$, as desired.