MATH 252 ASSIGNMENT 17 PROBLEM 8

ALEX FINK

Problem. Given $H \subseteq G$, we may view $F_k(H)$ as an $F_k(G)$-module via the ring homomorphism $\operatorname{res} : F_k(G) \to F_k(H)$. What does Frobenius Reciprocity say about $\operatorname{ind} : F_k(H) \to F_k(G)$ from this viewpoint? What can you conclude about the image of ind? Anything interesting you can say about the product of a monomial character with a linear character?

Solution. We may also view $F_k(G)$ as an $F_k(G)$-module in the canonical way. In this context, the statement of Frobenius reciprocity on class functions $\alpha \in F_k(G)$, $\beta \in F_k(H)$,

$$\operatorname{ind}(\operatorname{res}(\alpha)\beta) = \alpha \operatorname{ind}(\beta), \tag{1}$$

asserts simply that ind is an intertwining map for the actions of α on $F_k(G)$ and $F_k(H)$, given as multiplications by respectively $\operatorname{res}(\alpha)$ and $\operatorname{res}(\beta)$, i.e. that ind is a $F_k(G)$-module homomorphism.

Frobenius reciprocity also tells us that $\operatorname{ind}(F_k(H))$ is an ideal of the ring $F_k(G)$, for ind is certainly an additive homomorphism, and its image is closed under multiplication by arbitrary $\alpha \in \operatorname{ind}(F_k(H))$. As a ring $F_k(G) = \sum_{C \in \operatorname{cl}(G)} k_C$, where $\operatorname{cl}(G)$ denotes the set of all conjugacy classes of G, and $k_C = k$ with the projection given by evaluation at C (and, by finiteness of $\operatorname{cl}(G)$, it's insubstantial whether we use the direct product or direct sum here). An ideal of $F_k(G)$ is then a direct sum of ideals in its components k, and the only ideals of k are k and 0. To completely characterise $\operatorname{ind}(F_k(H))$, then, it remains only to be seen which classes of H are in the support of any induced class function. For this, in view of our formula

$$\operatorname{ind}(\beta)(g) = \frac{1}{|H|} \sum_{s \in G} \dot{\beta}(g^s)$$

we see that any class of G falling outside every conjugate of H is sent to zero by every induced character, while taking $\beta = 1_H$ we see that no other class has this property. So $\operatorname{ind}(F_k(H))$ is the set of class functions supported on $\bigcup_{s \in G} H^s$.

The product of a monomial character and a linear character is monomial. If, in (1), α and β are both linear characters, then their product is $(\alpha_H \beta)^G$, in which α_H is itself a linear character and hence so is its product with β.