MATH 252 ASSIGNMENT 16 PROBLEM 7

ALEX FINK

Problem. Let $\chi \in \mathrm{Irr}(G)$ over a group G. If χ comes from a real representation, show that $s(\chi) = 1$. (The converse is true also, but its proof is a bit more challenging.)

Solution. Let V be the vector space yielding χ. Our construction of a unitary form of χ, via averaging, can be carried out entirely within $\mathbb{R}$; do so and identify V with $\mathbb{R}^n$ in such a way that the resulting matricial form $D : G \to \mathrm{GL}_2(\mathbb{R})$ of χ is unitary. This identification induces an isomorphism $V \otimes V^* \cong \mathbb{M}_n(\mathbb{R})$ as vector spaces, sending the standard basis matrix E_{ij} with a single 1 in position (i, j) to $e_i \otimes \langle \cdot, e_j \rangle$.

In matricial terms, the action of $g \in G$ on the left factor of $V \otimes V^*$ is easily seen to be given by $A \mapsto D(g)A$, and the action on the right factor by $A \mapsto D(g^{-1})^t$, in which by unitarity and realness $D(g^{-1})^t = \overline{D(g)^t} = D(g)^t$.

We'll actually endow $V \otimes V^*$ with the diagonal G-module structure, so that the action of G is given by $g \cdot A = D(g)AD(g)^t$. Let M be the matrix corresponding to

$$(\sum_{g \in G} g) \cdot (\sum_i e_i \otimes \langle \cdot, e_i \rangle) \in V \otimes V^*,$$

so that $M = \sum_{g \in G} D(g)ID(g)^t$. Then M is symmetric; G-invariant, by the standard averaging argument; and positive definite, since for nonzero $v \in \mathbb{C}^n$, $v^\dagger M v = \sum_g (v^\dagger D(g))(D(g)^\dagger v)$ is a sum of positive squares, not all zero. In particular M is nonzero.

Next, since $V^* \cong V$ (noncanonically), we have $V \otimes V^* \cong V^2$. We have thus found a nonzero G-invariant symmetric tensor in V^2. Therefore its span gives a copy of the trivial $\mathbb{C}G$-module within $\mathrm{Sym}^2(V)$. By our discussion of representations of type 1, $s(\chi) = 1$, as desired.