MATH 252 ASSIGNMENT 15 PROBLEM 3

ALEX FINK

Problem. Let U, V, W be simple kG-modules, where k is a splitting field of characteristic 0 for G. Let T be the regular module kG.

(a) Show that $U \otimes T \cong (\dim U) \cdot T$.
(b) Show that $U \otimes V$ can be embedded in T.
(c) Show that W appears in $U \otimes V$ at most $\dim W$ times.

Solution.

(a) In characteristic zero every finite-dimensional kG-module is semisimple and so determined up to isomorphism by its irreducible direct summands, the list of which can be recovered from the character. So to show $U \otimes T \cong (\dim U) \cdot T$ it suffices to show the corresponding characters are equal. But $\chi_T(g) = n\delta_{g1}$, where $n = |G|$; so $\chi_{U\otimes T} = \chi_U \chi_T$ and $\chi_{(\dim U)T} = (\dim U)\chi_T$ are both zero on $G \setminus 1$, and they both take value $n\chi_U(1)$ at 1, so the characters are equal.

(b) This is a consequence of part (c). Given any simple kG-module W, we know that W occurs as a component of T exactly $\dim W$ times, and by (c) it occurs no more often in $U \otimes V$. Thus the simple summands of $U \otimes V$ are some subcollection of the simple summands of T, so that an embedding can be established carrying $U \otimes V$ to the sum of such a subcollection of the simple summands of T.

(c) Using our splitting field assumption, we wish to show that $k := [\chi_U \chi_V, \chi_W] \leq \dim W$. We have that

$$\begin{aligned} k &= [\chi_U \chi_V, \chi_W] \\ &= |G|^{-1} \sum_{g\in G} \chi_U(g)\chi_V(g)\chi_W(g^{-1}) \\ &= |G|^{-1} \sum_{g\in G} \chi_U(g)\chi_{V^*}(g^{-1})\chi_W(g^{-1}) \\ &= [\chi_U, \chi_{V^*}\chi_W], \end{aligned}$$

where, recall, V^* denotes the module contragredient to V. That is, k gives the number of times U appears in $V^* \otimes W$. Since the sum of the dimensions of a collection of simple summands of a kG-module M can't exceed $\dim M$, we have $k \dim U \leq \dim(V^*) \dim W = \dim V \dim W$. Likewise, $k = [\chi_V, \chi_{U^*}\chi_W]$, and this gives rise to $k \dim V \leq \dim U \dim W$. Multiplying, we have $k^2 \dim U \dim V \leq \dim U \dim V \dim W^2$, i.e. $k \leq \dim W$, as desired.