

MATH 252 ASSIGNMENT 15 PROBLEM 2

ALEX FINK

Problem. For $\chi, \chi_1, \dots, \chi_m \in \text{Irr}(G)$ (any G), show that TFAE:

- (a) $\ker \chi_1 \cap \dots \cap \ker \chi_m \subseteq \ker \chi$.
- (b) χ appears in some product $\chi_1^{i_1} \dots \chi_m^{i_m}$.

Show that (a), (b) are implied by but not equivalent to:

- (c) $\chi \in \mathbb{Z}[\chi_1, \dots, \chi_m]$ (the subring of $\text{Ch}(G)$ generated by $\{\chi_i\}$).

Solution. Write $K := \bigcap_{i=1}^m \ker \chi_i$.

[(b) \implies (a)] Express the product $\phi := \chi_1^{i_1} \dots \chi_m^{i_m}$ as a linear combination of irreducibles $\phi = \sum_{s=1}^s c_s \chi_s$, $c_i \in \mathbb{Z}$ (for notational convenience labelling the irreducibles so that the first m are the ones we've selected). For any $g \in K$ we have $\phi(g) = \phi(1)$; but by the triangle inequality

$$\phi(g) = |\phi(g)| \leq \sum_{i=1}^s c_s |\phi(g)| \leq \sum_{i=1}^s c_s \phi(1).$$

Since this is an equality, all the inequalities $c_s |\phi(g)| \leq c_s \phi(1)$ must be tight, so that when $c_s \neq 0$ we get $\phi(g) = \phi(1)$, i.e. $g \in \ker \chi_s$, for every $g \in K$.

[(a) \implies (b)]¹ We first handle the case in which every χ_i is linear, without an irreducibility hypothesis on χ but only the hypothesis that $\chi \neq 0$. Every $\chi_1^{i_1} \dots \chi_m^{i_m}$ is then a multiplicative character factoring through G/K , and the set of all these products is some finite abelian group C . Choose characters $\chi'_1, \dots, \chi'_l \in C$ such that $C = \bigoplus_{i=1}^l \langle \chi'_i \rangle$. Now

$$[G : K] = [G : \bigcap_i \ker \chi'_i] \leq \prod_i [G : \ker \chi'_i] = \prod_i o(\chi'_i) = |C| \leq |G/K| = [G : K],$$

so both inequalities here are equalities and C contains all linear characters on G/K , and indeed all irreducibles.

Assume towards a contradiction that $\langle \chi, \chi' \rangle = 0$ for all $\chi' \in C$. Since the character table of G/K is invertible, a suitable linear combination of these equations $\langle \chi, \chi' \rangle = 0$ expands to $\sum_{g \in K} \chi(g) = 0$. But by (a), $\chi(g) = \chi(1)$ for all $g \in K$, so that this implies $\chi(1) = 0$, contradiction.

Now, return to the general case that the χ_i are general irreducible characters. Let $H = \bigcap_{i=1}^m Z(i)$; then each $(\chi_i)_H$ is a scalar multiple of a linear character, so our special case above establishes the result on the restriction of all our characters to H . Fix exponents j_1, \dots, j_m such that for $\phi = \chi_1^{j_1} \dots \chi_m^{j_m}$, we have $\langle \chi_H, \phi_H \rangle > 0$. Next, for each $i = 1, \dots, m$, let $o_i > 0$ be an integer such that $\chi_i(g)^{o_i} = \chi_i(1)^{o_i}$ for all $g \in H$, and consider the character $\psi = \chi_1^{o_1} \dots \chi_m^{o_m}$. We have that $\ker \psi \supseteq H$, while for any $g \in G \setminus H$, $|\psi(g)| < \psi(1)$ strictly since an analogous strict inequality holds of some such i . Thus, for $k \gg 0$, we can arrange given any $\epsilon > 0$ that

¹This can't be the cleanest way to do this...

$|\psi^k(g)|/\psi^k(1) \leq \epsilon$ for all $g \in H \setminus G$. Hence we can arrange that $\langle \chi, \psi^k \phi \rangle > 0$: expanding this to $|G|^{-1} \sum_g \chi(g) \overline{(\psi^k \phi)(g)}$, the sum over only $g \in H$ will be positive since $\psi^k \phi$ agrees with ϕ on H , and the other terms can't change this for k sufficiently large.

[(c) \implies (b)] If instead we had $\langle \chi, \chi_1^{i_1} \dots \chi_m^{i_m} \rangle = 0$ for any tuple of indices i_1, \dots, i_m , then since these monomials are a \mathbb{Z} -generating set for $\mathbb{Z}[\chi_1, \dots, \chi_m]$ we'd have $\langle \chi, \psi \rangle = 0$ for any ψ in this ring, in particular $\psi = \chi$, contradiction.

[(b) $\not\Rightarrow$ (c)] As a counterexample consider $G = Q_8$ with classes ordered $(1, -1, i, j, k)$, $m = 1$, $\chi_1 = (2, -2, 0, 0, 0)$, $\chi = (1, 1, 1, -1, -1)$. It's apparent that $\ker \chi_1 \subseteq \ker \chi$; however, $\chi \notin \mathbb{Z}[\chi_1]$ since every element of $\mathbb{Z}[\chi_1]$ is equal on i and j , which χ is not.