Exercise 7: Let $G = K \rtimes H$ be as above (i.e. G is a Frobenius group in the usual notation), and let $1 \neq h \in H$. Show that the map $\varphi : K \to K$ defined by $\varphi(x) = x^{-1}x^h$ ($x \in K$) is a bijection.

Solution: First note that $\varphi(x) \in K$ because K is a normal subgroup, so $x^h \in K$ and $x^{-1} \in K$ if $x \in K$.

Since K is a finite group, to show φ is a bijection it suffices to show injectivity. Set $x, y \in K$ s.t. $\varphi(x) = \varphi(y)$, that is $y^{-1}y^h = y^{-1}h^{-1}yh = x^{-1}h^{-1}xh = x^{-1}x^h$. Equivalently:

$$y^{-1}h^{-1}yh = x^{-1}h^{-1}xh \iff h^{-1}y = yx^{-1}h^{-1}x \iff h^{-1}(yx^{-1})h = yx^{-1} \ .$$

Therefore, we have $w = yx^{-1} \in K$ which is fixed by conjugation by an element $h \neq 1, h \in H$. We want to show that $w = 1 \in G$. But we know (by corollary of Big Frobenius Thm) that H acts semiregularly on K^*, so in particular, w cannot be a fixed point of conjugation by $h \neq 1$ unless $w = 1$. Therefore, $w = 1$, and so $y = x$. Thus, φ is injective, and so it is a bijection by cardinality arguments. □