Exercise 6: For $N \subseteq H \subseteq G$ as in (5) (i.e. $N \lhd G$), let $\bar{G} = G/N$, and $\bar{H} = H/N$. For any $k\bar{H}$-module $\bar{V}$, let V denote $\bar{V}$ viewes as a kH-module (by pullback). Show that V^g is kG-isomorphic to the pullback of $\bar{V}^G$.

Solution: Since $\bar{V}$ is a $k\bar{H}$-module, then V has trivial N-action. Likewise $\bar{V}^{\bar{G}}$ is a $\bar{G}$-module and so we can view it as a kG-module with trivial N-action. By Exercise XVIII-6 we know that V^G has trivial N-action. So there is a chance for both kG-modules of being isomorphic. In fact, the natural map will do the trick.

Set $\psi : V^G \to \bar{V}^{\bar{G}}$ defined as $\psi(g \otimes v) = \bar{g} \otimes v$ and extend it k-linearly. As a remark, nota that in the (RHS) of the last expression, we are viewing $v \in \bar{V}$ (i.e., we forget about the H-action and only view the $\bar{H}$-action). We need to show that the map is well defined, i.e. we need to check the middle linearity:

$$\psi(g(h) \otimes v) = \overline{gh} \otimes v = \bar{g} \underbrace{\bar{h}}_{\in \bar{H}} \otimes_{k\bar{H}} v = \bar{g} \otimes (\bar{h} \cdot v) = \bar{g} \otimes (h \cdot v) = \psi(g \otimes (h \cdot v)) \ ,$$

since the action in V is given by the action in $\bar{V}$. Therefore, it is well-defined. It is clear that it is a kG-module homomorphism.

Let us show the surjectivity of ψ. Since it is k-linear, and a system of generators of $\bar{V}^{\bar{G}}$ as a k-vector space is given by $\{\bar{g} \times v : \bar{g} \in \overline{G}, v \in \bar{V}\}$, ψ is clearly surjective.

To finish, we only need to prove that both k-vector spaces have the same (finite) dimension. And we have $\dim_k V^G = [G : H] \dim_k V = [G : H] \dim_k \bar{V}$, whereas $\dim_k(\bar{V}^{\bar{G}}) = [\bar{G} : \bar{H}] \dim_k \bar{V} = \frac{[G:N]}{[H:N]} \dim_k \bar{V} = [G : H] \dim_k \bar{V}$, thus $\dim_k V^G = \dim_k(\bar{V}^{\bar{G}})$ as we desired. $\square$