Exercise 5: Let $N \subseteq H \subseteq G$ be groups, with $N \lhd G$. If N acts trivially on a kH-module V, show that N also acts trivially on the induced module V^G.

Solution: Denote $[G : N] = m$. To solve the exercise we will need to use the the block permutation matrix associated to the representation D^G given that $D : H \to \mathrm{Gl}_n(k)$ is the representation afforded by V. Recall the construction of D^G. Suppose V has a k-basis $\{e_1, \ldots, e_n\}$. An ordered k-basis for V^G would be $\{g_1 \otimes e_1, g_1 \otimes e_2, \ldots, g_1 \otimes e_n, g_2 \otimes e_1, \ldots, g_m \otimes e_n\}$, where $\{g_1, \ldots, g_m\}$ are a complete list of representatives of the left-cosets G/H. Using this ordered basis, we get:

$$D^G(g) = \begin{pmatrix} \dot{D}(g_1^{-1}gg_1) & \dot{D}(g_1^{-1}gg_2) & \ldots & \dot{D}(g_1^{-1}gg_m) \\ \vdots & \vdots & & \vdots \\ \dot{D}(g_m^{-1}gg_1) & \dot{D}(g_m^{-1}gg_2) & \ldots & \dot{D}(g_m^{-1}gg_m) \end{pmatrix} \in \mathbb{M}_{mn}(k)$$

and each block $\dot{D}(g_i^{-1}gg_j) \in \mathbb{M}_n(k)$. Denote each block entry of $D^G(g)$ by indices i, j where $i, j = 1, \ldots, m$.

To show that N acts trivially on V^G, we need to show that $D^G(h) = I_{nm}$ for all $h \in N$. Using our notation, this is equivalent to showing $\dot{D}(g_i^{-1}gg_j) = \delta_{ij} I_n$ for all $i, j = 1, \ldots, m$.

Suppose $i = j$. Since $N \lhd G$, we have $g_i^{-1}gg_i \in N \subseteq H$ if $g \in N$, so $\dot{D}(g_i^{-1}gg_i) = D(g_i^{-1}gg_i) = I_n$.

On the contrary, suppose $i \neq j$. We want to know if $(g_i^{-1}gg_j) \in H$ for some $g \in N$. But we have $g_i^{-1}gg_j = \underbrace{(g_i^{-1}gg_i)}_{\in N \subseteq H}(g_i^{-1}g_j)$, so $(g_i^{-1}gg_j) \in H$ iff $(g_i^{-1}g_j) \in H$. and this cannot happen if $i \neq j$ by construction of the left coset representativs $\{g_l\}$.

Thus $D^G(h) = I_{nm}$ if $h \in N$, so N acts trivially on V^G. $\square$