Exercise 4: Given $H \subseteq G$, we may view $F_k(H)$ as an $F_k(G)$-module via the ring homomorphism $res : F_k(G) \to F_k(H)$. What does Frobenius Reciprocity say about $ind : F_k(G) \to F_k(G)$ from this viewpoint? What can you conclude about the image of "ind"? Anything interesting you can say about the product of a monomial character with a linear character?

Solution: Set $\mu \in F_k(G)$ and $\nu \in F_k(H)$. The third form of Frobenius Reciprocity says that $(\nu\mu_H)^G = \nu^G\mu$. In our new notation, this will be $ind(\nu\, red(\mu)) = ind(\nu)\mu$, so if we use the $F_k(G)$-(right) module structure on $F_k(H)$, we get

$$ind(\nu \cdot \mu) = ind(\mu) \cdot \mu$$

so $ind : F_k(H) \to F_k(G)$ is an $F_k(G)$-module homomorphism.

Therefore, in particular, $ind(F_k(G))$ is a $F_k(G)$-submodule of $F_k(G)$.

Now suppose ν is a linear character on H and μ is a linear character. Using the previous viewpoint, we have $\nu^G \cdot \mu = (\nu\mu_H)^G$ and ν, μ_H are linear characters on H, so $(\nu\mu_H)$ is also a linear character on H. So, in particular, a product of a monomial character (ν^G) with a linear character (μ) is again a monomial character $((\nu\mu_H)^G)$. Thus, the k-vector space generated by the monomial characters is invariant under the action of linear characters. So it is a module over the k-algebra generated by the linear characters. $\square$