Exercise 3: We have skipped some of the formal proofs for the laws of induction for <u>class functions</u>. Supply a proof for one of the following:

1. $(\nu^E)^G = \nu^G$ where $H \subseteq E \subseteq G$;
2. $[\nu^G, \mu]_G = [\nu, \mu_H]$.

(Of course you are welcome to do both!)

Solution:

1. We will use the formulas of induction for class functions on G and E:

$$(\nu^E)^G(g) = \frac{1}{|E|}\sum_{t\in G} \dot{\nu}^E(g^t) \ .$$

And likewise, if $g^t \in E$

$$\nu^E(g^t) = \frac{1}{|H|}\sum_{s\in E} \dot{\nu}((g^t)^s) = \frac{1}{|H|}\sum_{s\in E} \dot{\nu}(g^{ts}) \ .$$

So:

$$(\nu^E)^G(g) = \frac{1}{|E|}\sum_{t:g^t\in E}\{\frac{1}{|H|}\sum_{s\in E}\dot{(\nu)}(g^{ts})\} = \frac{1}{|E|}\big(\frac{1}{|H|}\sum_{t:g^t\in E}\{\sum_{s\in E}\dot{\nu}(g^{ts})\}\big) =$$
$$= \frac{1}{|H|}\big(\frac{1}{|E|}\sum_{s\in E}\{\sum_{\substack{t\in G\\ g^t\in E}}\dot{\nu}(g^{ts})\}\big) \ .$$

The key-point is the following. Since $H \subseteq E \subseteq G$ and $\dot{\nu}(g^{ts}) = 0$ if $g^{ts} \notin H$, we can extend the second sum for $t \in G$, eliminating the extra condition $g^t \in E$. This is consequence of $g^{ts} = s^{-1}(t^{-1}gt)s \in H$ iff $g^t \in sHs^{-1}$. and $s \in E$, so $g^t \in sHs^{-1} \subseteq E$ implies $g^t \in E$. Hence,

$$(\nu^E)^G(g) = \frac{1}{|H|}\big(\frac{1}{|E|}\sum_{s\in E}\{\sum_{t\in G}\dot{\nu}(g^{ts})\}\big) \ .$$

Let us workout the expression $\frac{1}{|E|}\sum_{s\in E}\{\sum_{t\in G}\dot{\nu}(g^{ts})\}$. Consider the left-coset $G/E = \{t_1, \dots, t_m\}$, so each $t = t_i e$ for a unique pair $e \in E$ and $i = 1, \dots, m$. Hence,

$$\frac{1}{|E|}\sum_{s\in E}\{\sum_{t\in G}\dot{\nu}(g^{ts})\} = \frac{1}{|E|}\sum_{s\in E}\{\sum_{e\in E}\sum_{i=1}^{r}\dot{\nu}(g^{t_i(es)})\} \qquad (*)$$

And since we have $e, s \in E$, and multiplication by a fixed element is a bijection in E, we can change the index $e \in E$ to $e' = es \in E$. Hence:

$$(*) = \frac{1}{|E|}\sum_{s\in E}\{\sum_{e\in E}\sum_{i=1}^{r}\dot{\nu}(g^{t_i e})\} = \frac{1}{|E|}|E|\{\sum_{e\in E}\sum_{i=1}^{r}\dot{\nu}(g^{t_i e})\} = \sum_{t\in G}\dot{\nu}(g^t)$$

Therefore,

$$(\nu^E)^G(g) = \frac{1}{|H|}\sum_{t\in G}\dot{\nu}(g^t) = \nu^G(g) \ ,$$

for all $g \in G$, so $(\nu^E)^G = \nu^G$. $\square$

2. Combining the definitions of the bracket product and the induced class function we get:

$$[\nu^G,\mu]_G = \frac{1}{|G|}\sum_{g\in G}\nu^G(g^{-1})\mu(g) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{|H|}\sum_{t\in G}\dot{\nu}((g^{-1})^t)\mu(g^{-1}) =$$
$$= \frac{1}{|H|}\Big(\frac{1}{|G|}\sum_{t\in G}\sum_{g\in G}\dot{\nu}((g^{-1})^t)\mu(g)\Big) = \frac{1}{|H|}\Big(\frac{1}{|G|}\sum_{t\in G}\sum_{g\in G}\dot{\nu}((g^t)^{-1})\mu(g)\Big)\ .$$

Let us workout the expression $\frac{1}{|G|}\sum\limits_{t\in G}\sum\limits_{g\in G}\dot{\nu}((g^t)^{-1})\mu(g)$. At first, note that conjugating by a fixed t gives a bijection on G: $g \leftrightarrow g^t$. Thus, we can change the index g in the second sum by g^t. Hence:

$$\frac{1}{|G|}\sum_{t\in G}\sum_{g\in G}\dot{\nu}((g^t)^{-1})\mu(g) = \frac{1}{|G|}\sum_{t\in G}\sum_{g\in G}\dot{\nu}((g)^{-1})\mu(g^{t^{-1}})\ . \qquad (**)$$

Now we change t by t^{-1} in the first sum and we change the order of both sums, so

$$(**) = \frac{1}{|G|}\sum_{t\in G}\sum_{g\in G}\dot{\nu}((g)^{-1})\mu(g^t) = \frac{1}{|G|}\sum_{g\in G}\sum_{t\in G}\dot{\nu}((g)^{-1})\mu(g^t) = \frac{1}{|G|}\sum_{g\in G}\dot{\nu}((g)^{-1})\sum_{t\in G}\mu(g^t) =$$
$$= \sum_{g\in G}\dot{\nu}((g)^{-1})\Big(\frac{1}{|G|}\sum_{t\in G}\mu(g^t)\Big) = \sum_{g\in H}\nu((g)^{-1})\Big(\frac{1}{|G|}\sum_{t\in G}\mu(g^t)\Big)\ ,$$

since $g^{-1}\in H$ iff $g\in H$. And now, simply note that μ is a class function so $\mu(g^t)=\mu(g)$ for all t. Hence:

$$\sum_{g\in H}\nu((g)^{-1})\Big(\frac{1}{|G|}\sum_{t\in G}\mu(g^t)\Big) = \sum_{g\in H}\nu((g)^{-1})\Big(\frac{1}{|G|}\sum_{t\in G}\mu(g)\Big) = \sum_{g\in H}\nu((g)^{-1})\mu(g)\ .$$

Finally, replace this last expression in the first equation describing $[\nu^G,\mu]_G$:

$$[\nu^G,\mu]_G = \frac{1}{|H|}\sum_{g\in H}\nu((g)^{-1})\mu(g) = \frac{1}{|H|}\sum_{g\in H}\nu((g)^{-1})\mu_H(g) = [\nu,\mu_H]_H\ ,$$

since $g\in H$ so $\mu(g)=\mu_H(g)$. $\square$