Exercise 2: For groups $H \subseteq G$, let $\nu \in F_k(H)$ where $\operatorname{char} k \nmid |G|$. For $g \in G$, show that $\nu^G(g) = |H|^{-1}|C_G(g)| \sum_{h \in H \cap C} \nu(h)$, where C denotes the conjugacy class of g in G.

Solution: Since we are in the good characteristic case, we have a formula for $\nu \in F_k(G)$ in terms of $\dot{\nu}$, namely

$$\nu^G(g) = \frac{1}{|H|} \sum_{t \in G} \dot{\nu}(g^t) \ .$$

Let us analize each summand. By definition, $\dot{\nu}(g^t) = 0$ if $g^t \notin H$. So the nonzero summands will be to ones with $t \in G$ s.t. $g^t \in H$, and since $g^t \in C$, we get $g^t \in H \cap C$. Now we need to check how many t's give the same $g^t = h \in H \cap C$. We have

$$g^t = g^s \iff t^{-1}gt = s^{-1}gs \iff (ts^{-1})^{-1}g(ts^{-1}) = st^{-1}g(ts^{-1}) = g \iff ts^{-1} \in C_G(g) \ ,$$

so each $h \in H \cap C$ must be counted $|C_G(g)|$ times. Hence, the formula is:

$$\nu^G(g) = \frac{1}{|H|} \sum_{t \in G} \dot{\nu}(g^t) = \frac{1}{|H|} |C_G(g)| \sum_{h \in H \cap C} \nu(h) \ ,$$

as we wanted to show. $\square$