Exercise 1: (For fun-loving folks!) A group G is called funny if it has n irreducible $\mathbb{C}$-characters with degrees $1, 2, 3, \ldots, n$. Show that all funny groups are trivial.

Solution: We want to show that $m = 1$. By the Master Thm, we have $i|m$ for all $i = 1, \ldots, n$. Suppose $m > 1$. By the Magic Equation, we have $m = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, so $n > 1$. Hence, $6m = n(n+1)(2n+1)$ and $i \mid m$ for all i. In particular, $n-1 \mid m$, so $n-1 \mid n(n+1)(2n+1)$. Since $n-1$ and n are coprime, we get $n-1 \mid (n+1)(2n+1)$.

If n is even, then $n-1$ is odd and since $\gcd(n-1, n+1) \mid 2$, we get $\gcd(n-1, n+1) = 1$, so $n-1 \mid (2n-1)$. Since $n-1 \mid 2(n-1) = 2n-2$, we get $n-1 \mid (2n-1) - 2(n-1) = -3$, so $n-1 = 1, 3$. Thus $n = 2, 4$. In the first case, we get $m = 5$ and $2 \mid 5$, a contradiction. In the second case, $m = 30$ and $4 \mid 30$, a contradiction.

Suppose now n is odd, then $\gcd(n-1, n+1) = 2$ and so $\frac{n-1}{2} \mid (2n-1)$. Again, we get $\frac{n-1}{2} \mid (2n-1) - 2(n-1) = -3$ so $\frac{n-1}{2} = 1, 3$, or equiv $n-1 = 2, 6$. Hence, $n = 3, 7$. The first case yields $m = 14$ and $3 \mid 14$, whereas in the second case, $m = 7 \cdot 4 \cdot 5 = 140$ and $3 \mid m$ $(3 < 7 = n)$, and both situations cannot occur.

Thus, $m = 1$ and so G is trivial. $\square$