Exercise 8: Let G be any finite group, and let $\Gamma = G \times G$. Let $U = kG$ be viewed as a $k\Gamma$-module via the action $(g_1, g_2) \cdot \alpha = g_1 \alpha g_2^{-1}$. Show that U is a monomial module over $k\Gamma$.

Solution: Let us describe the situation: we have two finite groups G, Γ and a morphism between them $\Delta : G \hookrightarrow \Gamma$, $\Delta(g) = (g, g)$.

Since $U = kG$ has dimension $n = |G|$ and $[\Gamma : G] = n$, there is a chance that the $k\Gamma$-module structure of U comes from an induced structure for a kG-module V. We set $V = k$, wich is a kG-module with trivial action over G. In this case $\dim_k V = 1$, with basis $\{1\}$.

By definition, $V^\Gamma = k\Gamma \otimes_{kG} V$ with structure given by $\alpha(\beta \otimes v) = (\alpha\beta) \otimes v$ and the so called middle linearity: $\alpha \cdot (g, g) \otimes v = \alpha \otimes gv$ $(\alpha, \beta \in \Gamma, g \in G, v \in V)$.

We define the map $\psi : V^\Gamma \to U$ as $\psi((g_1, g_2) \otimes a) = g_1 a g_2^{-1} \in U$ and we extend it by k-linearity. We need to show that it is well-defined and that it is a morphism of $k\Gamma$-modules. To check the well-defined condition, we need to verify the middle linearity:

$$\begin{aligned}\psi((g_i, g_i')) \cdot \Delta(g) \otimes a) &= \psi((g_i, g_i')(g, g) \otimes a) = \psi((g_i g, g_i' g) \otimes a) = (g_i g) a (g_i' g)^{-1} = \\ &= g_i (g a g^{-1})(g_i')^{-1} = \psi((g_i, g_i') \otimes g a g^{-1}) = \psi((g_i, g_i') \otimes a) = \psi((g_i, g_i') \otimes (g \cdot a))\end{aligned}$$

where $a \in k = V$, $g_i, g_i', g \in G$.

To show that it is a $k\Gamma$-linear map, it suffices to show the Γ-linearity, since the map is clearly k-linear by construction. We have $\psi((g_1', g_2') \cdot ((g_1, g_2) \otimes a)) = \psi(g_1' g_1, g_2' g_2) \otimes a) = g_1' g_1 a (g_2' g_2)^{-1} = g_1' (g_1 a g_2^{-1})(g_2')^{-1} = (g_1', g_2') \cdot \psi((g_1, g_2) \otimes a)$.

To finish, we need to show that the map is an isomorphism. For this, since it is k-linear and both spaces hace the same finite dimension over k, it suffices to show that it is injective or surjective. Let us prove the latter, by showing that a k-basis of U (namely G) lies in the image of ψ. In this case, this follows immediately by picking $a = 1 = g_2$ and $g_1 = g$, so $\psi((g, 1) \otimes 1) = g \in G$. By k-linearity, $\operatorname{im} \psi \subset k\langle g : g \in G \rangle = kG$, so the map is surjective, and hence an isomorphism. Therefore $U \simeq V^\Gamma$ is a monomial module over $k\Gamma$. $\square$