Exercise 6: Let $H \subseteq G$, and let V be a kH-module. For $g \in G$, note that $g \cdot V = g \otimes V$ in the induced module $V^G = kG \otimes_{kH} V$. Show that $g \cdot V$ is a $k[H^{g^{-1}}]$-module, and check that $(g \cdot V)^G \simeq V^G$.

Solution:

We want to show that $g \cdot V = g \otimes V$ is a $k[H^{g^{-1}}]$-module. We know that it is a k-vector space and $g \cdot V \subset V^G$, and V^G is a kG-module, hence a $k][$-module ($H^{g^{-1}} \subset G$ is a subgroup since que are conjugating w.r.t. a unique element). Therefore, we only need to check that $(gHg^{-1}) \cdot (g \cdot V) \subset g \cdot V$. Pick $h \in H, v \in V$, then $(ghg^{-1}) \cdot (g \cdot v) = (ghg{-}1g) \otimes v = (gh) \otimes v = g \otimes hv \in g \cdot V$ by middle-linearity and the fact that V is a kH-module.

We want to show that $(g \cdot V)^G \simeq V^G$ as kG-modules. For this, we note that both spaces have the same finite dimension $[G : H] \dim_k V = \frac{|G|}{|H|} \dim_k V = \frac{|G|}{|H^{g^{-1}}|} \dim_k (g \cdot V)$ over k. Thus, we only need to construct a kG-module homomorphism between these spaces and show that it is surjective.

We define $\psi : (g \cdot V)^G \to V^G$ as follows $\psi(g' \otimes_{k[H^{g^{-1}}]} (g \cdot v)) = (g'g) \otimes_H v$ and we extend it k-linearly. We need to show this is well-define, i.e. check middle-linearity:

$$\psi((g'(ghg^{-1})) \otimes (g \cdot v)) = g'(ghg^{-1})g \otimes_{kH} v = g'gh \otimes_{kH} v = (g'g) \otimes_{kH} (h \cdot v) = (g'g) \otimes_{kH} (hv) = \\ = \psi(g' \otimes (g \cdot \underbrace{hv}_{\in V})) = \psi(g' \otimes (gh) \cdot v)) = \psi(g' \otimes (ghg^{-1})(g \cdot v)) \, .$$

Next, we prove that ψ is a morphism of kG-modules. Since it is k-linear by construction, we need to show that it behaves well under the G-action. Namely, $\psi(g' \cdot (g'' \otimes_{k[H^{g^{-1}}]} (g \cdot v))) = \psi((g'g'') \otimes_{k[H^{g^{-1}}]} (g \cdot v)) = (g'g''g) \otimes v = g' \cdot (g''g \otimes v) = g' \cdot \psi(g'' \otimes_{k[H^{g^{-1}}]} (g \cdot v))$ (where $g', g'' \in G$, $v \in V$).

To finish, we show that the map ψ contains a k-basis of V^G on its image, namely $\{g' \otimes v : g' \in G, v \in V\}$. For this, consider $g'' = g'g^{-1}$, so $\psi(g'' \otimes (g \cdot v)) = g''g \otimes v = g' \otimes v$. Hence, since the map ψ is k-linear, we have that ψ is surjective, hence, an isomorphism by dimension arguments. $\square$