Exercise 5: Let $H \subseteq G$. If $D; H \to O_n(k) := \{M \in \mathbb{M}_n(k) | M^t M = I_n\}$ the an orthogonal representation, show that D^G is also an orthogonal representation. If $D : H \to U_n(\mathbb{C})$ is a unitary representation, show that D^G is also a unitary representation.

Solution: Suppose D is an orthogonal representation, that is $D(g)^t D(g) = I_n$. Set $m = [G : H]$. We know then that D^G corresponds to a mn-dimensional representation, and the matrix D^G is a blocked generalized permutation matrix. Moreover, if V is the kH-module represented by D, we have a k-basis $\{e_1, \ldots, e_n\}$ for V. An ordered k-basis for V^G would be $\{g_1 \otimes e_1, g_1 \otimes e_2, \ldots, g_1 \otimes e_n, g_2 \otimes e_1, \ldots, g_m \otimes e_n\}$, were $\{g_1, \ldots, g_m\}$ are a complete list of representatives of the left-cosets G/H. Thus, by our class discussions we have

$$D^G(g) = \begin{pmatrix} \dot{D}(g_1^{-1} g g_1) & \dot{D}(g_1^{-1} g g_2) & \ldots & \dot{D}(g_1^{-1} g g_m) \\ \vdots & \vdots & & \vdots \\ \dot{D}(g_m^{-1} g g_1) & \dot{D}(g_m^{-1} g g_2) & \ldots & \dot{D}(g_m^{-1} g g_m) \end{pmatrix} \in \mathbb{M}_{mn}(k)$$

and each block $\dot{D}(g_i^{-1} g g_j) \in \mathbb{M}_n(k)$. Denote each block entry of $D^G(g)$ by indices i, j where $i, j = 1, \ldots, m$.

Since all blocks have the same size $n \times n$, to multiply $D^G(g)^t$ and $D^G(g)$ we can do it in a "block fashion", namely $(D^G(g)^t D^G(g))_{i,j} = \sum_{l=1}^{m} (D^G(g)^t)_{i,l} D^G(g)_{l,j} \in \mathbb{M}_n(k)$. Now, by definition we have $\sum_{l=1}^{m} (D^G(g)^t)_{i,l} D^G(g)_{l,j} = \sum_{l=1}^{m} ((D^G(g))_{l,i})^t D^G(g)_{l,j} = \sum_{l=1}^{m} (\dot{D}(g_l^{-1} g g_i))^t \dot{D}(g_l^{-1} g g_j)$.

By definition, we have that the non-zero summand are those for which $g_l^{-1} g g_i \in H$ and $g_l^{-1} g g_j \in H$, so $\exists h, h' \in H$ s.t. $g_l = g g_i h = g g_j h'$. Therefore, in particular, $g_i h = g_j h'$, and so, by construction, we have $i = j$. Hence, $(D^G(g)^t D^G(g))_{i,j} = 0 \in \mathbb{M}_n(k)$ if $i \neq j$.

If $i = j$ we have that the elements g_l that give non-zero summands are of the form $g_l = g g_i h$ for some $h \in H$, and so there is a unique index l verifying this property, namely the one corresponding to the left coset $g g_i H$. So $g_l^{-1} g g_i = h \in H$ and thus $(D^G(g)^t D^G(g))_{i,i} = (\dot{D}(g_l^{-1} g g_i))^t \dot{D}(g_l^{-1} g g_i) = D(h)^t D(h) = I_n$ since D is an orthogonal representation.

Hence, we have that $D^G(g)^t D^G(g) = I_{nm}$ for all $g \in G$ so D^G is on orthogonal representation.

Concerning the case of a unitary representation, we need to show that $\overline{D^G(g)^t} D^G(g) = I_{nm}$. The arguments given above work in this case almost exactly, but we have to take into account the complex conjugation, and use the fact that $\overline{D(h)}^t D(h) = I_n$ for all $h \in H$. The fact that $\overline{\dot{(D)}(g)} = 0$ iff $\dot{(D)}(g) = 0$ guarantees that the previous counting argument works also in the unitary case. $\square$