Exercise 3: Let $G = \langle \sigma, \varphi | \sigma^7 = \varphi^3 = 1, \sigma^\varphi = \sigma^2 \rangle$ and $\zeta = e^{2\pi i/7}$. In Chap. 0, we have come up with the representations $D, D' : G \to \mathrm{Gl}_3(\mathbb{C})$, where $D(\sigma) = \mathrm{diag}(\zeta, \zeta^2, \zeta^4)$, $D'(\sigma) = (\zeta^6, \zeta^5, \zeta^3)$, and $D(\varphi) = D'(\varphi) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Show that both D, D' are monomial representations.

Solution: Since we have $\dim_{\mathbb{C}} D = \dim_{\mathbb{C}} D' = 3$, we need to find a subgroup $H \subset G$ with $[G : H] = 3$ and two 1-dim representations that induce D, D' over G. Our natural candidate for H is $H = \langle \sigma \rangle$, which is a normal subgroup of G. We know that all representations of H are 1-dimensional and, moreover, they verify $\chi_l(\sigma^j) = (\zeta^l)^j$ for all $l, j = 0, \ldots, 6$.

We want to see what do the induced representations χ_l^G look like. For this, we will use the block construction for χ_l^G in the basis $\{g_1 \otimes 1; g_2 \otimes 1; g_3 \otimes 1\}$ for $\{g_1, g_2, g_3\} = \{\varphi, \varphi^2, \varphi^3 = 1\} = G/H$, i.e. $g_i = \varphi^i$. Hence,

$$\chi_l^G(g) = \begin{pmatrix} \dot\chi_l(g_1^{-1} g g_1) & \dot\chi_l(g_1^{-1} g g_2) & \dot\chi_l(g_1^{-1} g g_3) \\ \dot\chi_l(g_2^{-1} g g_1) & \dot\chi_l(g_2^{-1} g g_2) & \dot\chi_l(g_2^{-1} g g_3) \\ \dot\chi_l(g_3^{-1} g g_1) & \dot\chi_l(g_3^{-1} g g_2) & \dot\chi_l(g_3^{-1} g g_3) \end{pmatrix} \in \mathbb{M}_3(\mathbb{C}) ,$$

and each block $\dot\chi_l(g_i^{-1} g g_j) \in \mathbb{M}_1(\mathbb{C}) = \mathbb{C}$. Denote each block entry of $\chi_l^G(g)$ by indices i, j where $i, j = 1, 2, 3$. Note that, since G is generated by σ and φ, we only need to characterize $\chi_l^G(\varphi)$ and $\chi_l^G(\sigma)$.

In this case, we have $g = \varphi^k \sigma^s$, so $g_i^{-1} g g_j = \varphi^{-i} \varphi^k \sigma^s \varphi^j = \varphi^{-i+k+j} \varphi^{-j} \sigma^s \varphi^j = \varphi^{-i+k+j} \sigma^{2^j s}$. Therefore, $(\chi_l^G(\varphi))_{i,j} = \dot\chi_l(\varphi^{1+j-i}) = 0$ if $i \neq 1 + j \pmod 3$ and it's $= 1$ if not, and $(\chi_l^G(\sigma))_{i,j} = \dot\chi_l(\varphi^{j-i} \sigma^{2^j}) = 0$ if $i \neq j$ and if $i = j$ we have $(\chi_l^G(\sigma))_{i,i} = \chi_l(\sigma^{2^i s}) = \zeta^{l 2^i}$. Hence:

$$\chi_l^G(\varphi) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad ; \quad \chi_l^G(\sigma) = \begin{pmatrix} \zeta^{2l} & 0 & 0 \\ 0 & \zeta^{4l} & 0 \\ 0 & 0 & \zeta^l \end{pmatrix} = \begin{pmatrix} \zeta^{2l} & 0 & 0 \\ 0 & (\zeta^{2l})^2 & 0 \\ 0 & 0 & (\zeta^{2l})^4 \end{pmatrix} .$$

Hence, to obtain D we as χ_l^G need to consider $2l = 1 \pmod 7$, that is $l = 4$, and for D' we need to take $2l = 6 \pmod 7$, that is $l = 3$. Therefore, D and D' are monomial representations. □