Exercise 2: Over $\mathbb{C}$, show that the irreducible 2-dim representations of the dihedral group $G = D_n$ are monomial. (In fact, they are all induced from linear characters on the normal subgroup H consisting of the n rotations in G.)

Solution: Let us recall the irreducible 2 dimensional representations over $\mathbb{C}$ of D_n given by Ex IV-10. We use the notation of the solution by "Dihedral Group Fan club".

If $n = 2m+1$ is odd, we have m irreducible 2-dim representations denoted by α_h $(1 \le h \le m)$:

	1	s	r	$\cdots$	r^j	$\cdots$	r^m
α_h	2	0	$2\cos(h\theta)$	$\cdots$	$2\cos(hj\theta)$	$\cdots$	$2\cos(hm\theta)$

where $\theta = 2\pi/n$.

In the even case, $n = 2m$ we have $m-1$ irred 2-dim representations also denoted by α_h $(1 \le h \le m-1)$:

	1	s	sr	r	$\cdots$	$r^j(j<m)$	$\cdots$	r^m
α_h	2	0	0	$2\cos(h\theta)$	$\cdots$	$2\cos(hj\theta)$	$\cdots$	$2(-1)^h$

where $\theta = 2\pi/n$.

We want to show that these representations are all induced from linear characters on the normal subgroup $H = \langle r \rangle \lhd G$. In fact, since $[G:H]1 = [G:H]\alpha_h(1) = \alpha_h^G(1) = 2$, then we know that $[G:H] = 2$ so in particular it is normal of order n, and the subgroup of all rotations verifies this condition.

We then need to describe the linear characters of $H = \langle r \rangle$ (i.e. cyclic subgroup of order n, but we know that the characters are given by the formulas $\chi_l(r^j) = (\zeta^l)^j$, where $\zeta = e^{2\pi i/n} = e^{\theta i/n}$ for $l = 0, \ldots, n-1$. Note that $\cos(lj\theta) = \mathrm{Re}((\zeta^l)^j)$.

Since $H \lhd G$, we have $\chi^G(G \setminus H) = 0$, so in this case $\chi^G(s) = 0$. And in the case of n odd, $\chi^G(sr) = 0$, which agrees with all α_h. To finish, we need to find for each h a suitable representation χ_l of H s.t. $\chi_l^G(r^j) = \alpha_h(r^j)$ for all j. For this, we will use the expression we deduced in class for the induced character, in terms of the conjugacy classes, namely: $\chi^G(g) = \frac{1}{|H|}\sum_{t\in G}\dot{\chi}(g^t)$ (note that we are in the good characteristic case).

In our case, if $\chi = \chi_l$, then since $H \lhd G$ and $r^j \in H$, we get $\chi_l^G(r^j) = \frac{1}{n}\sum_{t\in G}\dot{\chi}((r^j)^t) = \frac{1}{n}\sum_{t\in G}\chi((r^j)^t) = \frac{1}{n}\sum_{k=0}^{n-1}\{\chi((sr^k)^{-1}(r^j)(sr^k))+\chi(r^{-k}(r^j)r^k)\} = \frac{1}{n}\sum_{k=0}^{n-1}\{\chi(r^{-j})+\chi(r^j)\} = \frac{1}{n}n\{\overline{\chi(r^j)}+\chi(r^j)\} = \overline{\chi(r^j)} + \chi(r^j) = 2\mathrm{Re}(\zeta^{lj}) = 2\cos(lj\theta) = \alpha_l(r^j)$.

Note that in case $n = 2m$, this equality also holds for $j = m$, since $\chi_l^G(r^m) = 2\cos(lm\theta) = 2\cos(2lm\pi/2m) = 2\cos(l\pi) = 2(-1)^l = \alpha_l(r^m)$.

Therefore, $\alpha_h = (\chi_h)^G$ for all h in the corresponding interval depending on the parity of n. So α_h is a monomial representation. $\square$