Exercise 1:

Let $k = \mathbb{Q}$ and $G = \mathbb{A}_4$. Let χ be any one of the three nontrivial linear characters on the normal 2-Sylow group of G. Show that χ^G is the tetrahedral character on G (which is also the character of the 3-dim. reduced permutation representation of G on the G-set $\{1,2,3,4\}$).

Solution: We have $H = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} = \mathbb{K}_4 \lhd G = \mathbb{A}_4$ the normal 2-Sylow subgroup of G. We have that $[G : H] = 3$. We know that the three non-trivial characters on H have dimension 1 (H is abelian):

	1	a	b	ab
	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

where $a = (1\,2)(3\,4)$, $b = (1\,3)(2\,4)$. We what to compute χ_i^G for each $i = 2,3,4$. For this, we will use the formula derived for the extended character, since we are in the good characteristic case. Since $H \lhd \mathbb{A}_4$, we know that $\chi_i^G(\mathbb{A}_4 \setminus H) = 0$. So we need to compute the character over H:

$$\chi_i^G(h) = \frac{1}{|H|}\sum_{t\in\mathbb{A}_4} \dot{\chi}_i(\underbrace{t^{-1}ht}_{in H}) = \frac{1}{4}\sum_{t\in\mathbb{A}_4} \chi_i(t^{-1}ht) = \chi_i(h) + \chi_i((1\,2\,3)^{-1}h(1\,2\,3)) + \chi_i((1\,2\,4)^{-1}h(1\,2\,4)) =$$
$$= \chi_i(h) + \chi_i((1\,3\,2)h(1\,2\,3)) + \chi_i((1\,4\,2)h(1\,2\,4)) ,$$

since the left-cosets of H over G are given by $\{1, (1\,2\,3), (1\,2\,4)\}$. We have:

$$(1\,3\,2)a(1\,2\,3) = (1\,3)(2\,4) = b\ ;\ (1\,3\,2)b(1\,2\,3) = (1\,4)(2\,3) = ab;\ (1\,3\,2)ab(1\,2\,3) = bab = a\ ;$$
$$(1\,4\,2)a(1\,2\,4) = ab\ ;\ (1\,4\,2)b(1\,2\,4) = a\ ;\ (1\,4\,2)ab(1\,2\,4) = aba = b\ .$$

Therefore, in any case, we get $\chi_I^G(a) = \chi_i^G(b) = \chi_i^G(ab) = \sum_{\substack{h\in H\\ h\neq 1}} \chi_i(h)$, and $\chi_i^G(1) = 3$ for all $i = 2,3,4$. Hence:

	1	a	b	ab
	1	1	1	1
χ_2^G	3	-1	-1	-1
χ_3^G	3	-1	-1	-1
χ_4^G	3	-1	-1	-1

and $\chi_i^G(G \setminus H) = 0$. Thus, for all i, since a, b, ab lie in the conjugate class of a over G, we get $\chi_i^G = (3, -1, 0, 0)$ (i.e. $\chi_i^G(1) = 3$, $\chi_i^G(a) = -1$, $\chi_i^G((1\,2\,3)) = \chi_i^G((1\,2\,3))^2 = 0$), which corresponds to the tetrahedral representation of $\mathbb{A}_4$. $\square$