Exercise 9: Let $G = D_r$ be the dihedral group of order $n = 2r$, and write $r = 2m+1$ if r is odd, and $r = 2m$ if r is even. Show that the counting formula for the number t of involutions in G ($t = \sum_{\chi \neq 1} s(\chi)\chi(1)$) leads to $t = 2m+1$. (Check that this is the correct number by an explicit enumeration of the involutions.)
Referring to the bound $t^2 \leq (s-1)(n-1)$ (where $s = |\mathrm{Irr}(G)|$), show that the difference $(s-1)(n-1) - t^2$ is m when r is odd, and $3(m-1)$ when r is even!

Solution: Denote $D_r = \langle \tau, \sigma \mid o(\sigma) = 2; o(\tau) = r; \tau\sigma = \sigma\tau^{-1} \rangle$.

To use the counting formula for the number of involutions in G we need to compute the Frobenius-Schur indicators of the characters. By definition, $s(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \frac{1}{2r} \sum_{g \in G} \chi(g^2)$. As an important remark, we know By Ex IV-10 that the characters are real. Hence, the values of the indicators are ± 1.

To be able to compute the indicators, we need to know how the square of elements behave under the conjugacy action. Note that elements of the form $\sigma\tau^l$ square to 1, hence $\chi((\sigma\tau^l)^2) = \chi(1)$. Hence, it remains to compute $\chi((\tau^l)^2) = \chi(\tau^{2l})$. So we only need to check the conjugacy classes of elements of the form τ^{2l} for $l \in \mathbb{Z}_r$. By simple computation, we get $C(\tau^{2l}) = \{\tau^{2l}, \tau^{-2l}\}$, which has cardinality two iff $4l \neq 0 \pmod r$. Hence, we have two cases, depending on the parity of r.

If $r = 2m+1$ is odd, $|\tau^{2l}| = 2$ if $l \neq 0$ and we have $m+1$ conjugacy classes, whereas if $r = 2m$, then $|\tau^{2l}| = 1$ for $l = 0, m$ and $|\tau^{2l}| = 2$ otherwise, and we have $m+1$ conjugacy classes.

$$s(\chi) = \frac{1}{2r}(\sum_{l=0}^{r-1} \chi((\sigma\tau^l)^2) + \sum_{l=0}^{r-1} \chi((\tau^l)^2) = \frac{1}{2r}(\sum_{l=0}^{r-1} \chi(1) + \sum_{l=0}^{r-1} \chi(\tau^{2l}) = \frac{1}{2r}(r\chi(1) + \sum_{l=0}^{r-1} \chi(\tau^{2l}) \ .$$

Hence, we have:
- If $r = 2m+1$:

$$s(\chi) = \frac{1}{2r}(r\chi(1) + 2\sum_{l=1}^{m} \chi(\tau^{2l}) + \chi(1)) = \frac{1}{2r}((r+1)\chi(1) + 2\sum_{l=1}^{m} \chi(\tau^{2l})) \ .$$

- If $r = 2m$:

$$s(\chi) = \frac{1}{2r}(r\chi(1) + 2\sum_{l=1}^{m-1} \chi(\tau^{2l}) + \chi(1) + \chi(\tau^{2m})) = \frac{1}{2r}((r+2)\chi(1) + 2\sum_{l=1}^{m-1} \chi(\tau^{2l})) \ .$$

By Ex IV-10, we know the character table of D_r (we follow the notation of the resolution of this exercise by the "Dihedral Group Fan Club"), so we can compute the Frobenius-Shur indicator depending on the parity of r:
- If $r = 2m+1$, we have

$$s(\chi_2) = \frac{1}{2(2m+1)}(2(m+1)1 + 2\sum_{l=1}^{m} 1) = \frac{1}{2(2m+1)}(2(m+1) + 2m) = \frac{1}{2(2m+1)}(4m+2) = 1 \ ;$$

$$s(\alpha_h) = \frac{1}{2(2m+1)}(2(m+1)2 + 2\sum_{l=1}^{m} 2\cos(h2l\theta)) = \frac{1}{2(2m+1)}(4(m+1) + 4\sum_{l=1}^{m} \cos(h2l\theta)) \ ,$$

where $\theta = 2\pi/(2(2m+1))$ and $1 \leq h \leq m$. We know that $s(\alpha_h) = \pm 1$. And since $4(m+1) + 4\sum_{l=1}^{m} \cos(h2l\theta)) \geq 4(m+1) - 4\sum_{l=1}^{m}(-1) = 4m+4-4m = 4 > 0$ and $2(2m+1) > 0$, we get $s(\alpha_h) > 0$ for all h, hence $s(\alpha_h) = 1$ for all h.

Thus: $t = (1 \cdot 1 + \sum_{h=1}^{m} 2 \cdot 1) = 1 + 2m$.

• If $r = 2m$, we have

$$s(\chi_4) = \frac{1}{2(2m)}(2(m+1)1 + 2\sum_{l=1}^{m} 1) = \frac{1}{2(2m+1)}(2(m+1) + 2m) = \frac{1}{2(2m+1)}(4m+2) = 1\ ;$$

$$s(\chi_2) = s(\chi_3) = \frac{1}{2(2m)}((2m+2) \cdot 1 + 2\sum_{l=1}^{m-1}(-1)^{2l}) = \frac{1}{2(2m)}((2m+2) + 2\sum_{l=1}^{m-1} 1) =$$

$$= \frac{1}{2(2m)}((2m+2) + 2(m-1)) = \frac{4m}{4m} = 1\ ;$$

$$s(\alpha_h) = \frac{1}{4m}(2m+2)2 + 4\sum_{l=1}^{m-1}(cos(2lh\theta)) = \frac{1}{4m}(4(m+1) + 4\sum_{l=1}^{m-1} cos(2lh\theta))\ ,$$

where $\theta = 2\pi/(2(2m+1))$ and $1 \leq h \leq m-1$. By similar arguments as used before we get $s(\alpha_h) > 0$ hence $s(\alpha_h) = 1$.

Therefore: $t = (3 \cdot 1 + (m-1) \cdot 2 = 2m + 1$.

Let us compute explicitly the number of involutions, depending on the parity of r. We know that all symmetries $\sigma\tau^l$ with $0 \leq l \leq r-1$ are involutions. So it remains to count involutions of the form r^l, hence $2l \neq 0 \pmod r$ *but* $l \neq 0 \pmod r$.

Thus, ir r is odd $(2 : r) = 1$ and so the first equation leads to $l = 0 \pmod r$. Whereas if $r = 2m$ is even, we get $2l = 0 \pmod r$ iff $l = 0 \pmod m$, and so $l \neq 0 \pmod r$ leaves $l = m$ as the only possibility.

Hence, if r is odd, we have $t = r = 2m+1$, while $r = 2m$ even gives $t = r + 1 = 2m + 1$. And this agrees with our previous calculation of t using the counting formula.

To finish, let us compute the difference $(s-1)(n-1) - t^2$, according to the parity of r.

• If $r = 2m+1$, by Ex IV-10 we know that $s = 2 + m$ and $n = 2(2m+1)$, so $(s-1)(n-1) - t^2 = (m+1)(4m+1) - (2m+1)^2 = 4m^2 + 5m + 1 - (4m^2 + 1 + 4m) = m$.

• If $r = 2m$, we know $s = 4 + (m-1) = m + 3$, and $n = 2(2m) = 4m$, so $(s-1)(n-1) - t^2 = (m+2)(4m-1) - (2m+1)^2 = 4m^2 + 7m - 2 - (4m^2 + 1 + 4m) = 3m - 3 = 3(m-1)$. $\square$