

**Exercise 8:** Using the character table for the simple group  $G = \text{PSL}_2(\mathbb{F}_7)$ , compute the Frobenius-Schur indicator for all  $\chi \in \text{Irr}(G)$ .

**Solution:** Recall the character table of  $\text{PSL}_2(\mathbb{F}_7)$ :

	1	$g_1$	$g_2$	$g_3$	$g_4$	$g_5$
	1	21	42	56	24	24
$\chi_1$	1	1	1	1	1	1
$\chi_2$	3	-1	1	0	$\sigma$	$\bar{\sigma}$
$\chi_3$	3	-1	1	0	$\bar{\sigma}$	$\sigma$
$\chi_4$	6	2	0	0	-1	-1
$\chi_5$	7	-1	-1	1	0	0
$\chi_6$	8	0	0	-1	1	1

where  $\sigma = \frac{-1+i\sqrt{7}}{2} \notin \mathbb{R}$ .

By definition, we have  $s(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \frac{1}{168} \sum_{g \in G} \chi(g^2)$ , but this is useless in groups of even order. On the other hand we know that  $s(\chi)$  can have 3 values:  $0, 1, -1$ . On the other hand, if  $\chi_i$  is not real, then we know that  $s(\chi_i) = 0$ . Thus, in our case:  $s(\chi_i) = 0$  for  $i = 2, 3$ . And we also have  $s(\chi_1) = 1$ .

To compute the other indicators we will use the involutio count formula, namely  $t := \sum_{i=2}^6 s(\chi_i)\chi_i(1)$ , where  $t := \#\{\text{involutions of } \text{PSL}_2(\mathbb{F}_7)\}$ .

Thus, we need to compute the number of involutions on  $\text{PSL}_2(\mathbb{F}_7)$ . In this case, we get:

$$t = 3 \cdot 0 + 3 \cdot 0 + 6 \cdot s(\chi_4) + 7 \cdot s(\chi_5) + 8 \cdot s(\chi_6) .$$

On the other hand, note that since the remaining unknown indicators correspond to real characters, we know that the indicators equal  $\pm 1$ .

If we consider the maximal possible value for  $t$ , this corresponds to the sum  $6 + 7 + 8 = 21$ . And by taking a look at the character table, this is the least cardinality for a non-trivial conjugacy class (the only trivial conjugacy class is  $C(1)$ ). Since the order of  $G$  is even, we know that it has an involution (by Cauchy's Thm). Hence, the cardinality of it conjugacy class gives a lower bound for the number of involutions (because all elements in  $C(g)$  have order  $o(g)$ ). Since the only conjugacy class with one element is  $C(1)$ , then  $t \geq 21$ . Thus, we get  $t = 21$  and the only possible choice for the corresponding indicators is  $s(\chi_4) = s(\chi_5) = s(\chi_6) = 1$ .

Hence:

$$s(1_G) = s(\chi_4) = s(\chi_5) = s(\chi_6) = 1 ; s(\chi_2) = s(\chi_3) = 0 .$$