Exercise 5: Let G be a finite nonabelian simple group. Show that any subgroup $H \subseteq G$ of prime-power index is centerless (that is, $\mathcal{Z}(H) = \{1\}$). Deduce from this that G has no abelian subgroup of prime-power index.

Solution: Assume the contrary, and let $H \subseteq G$ be a subgroup of G of prime-power index. Since G is nonabelian and simple, $\mathcal{Z}(G) = \{1\}$. Thus, $H \subsetneq G$ and so $p \mid [G : H]$.

Let $h \neq 1$, $h \in \mathcal{Z}(H)$. Then $C_G(h) \supseteq H$. And by Lagrange's Thm, $|H| \mid |C_G(H)|$. Therefore, $|h| = \frac{|G|}{|C_G(h)|} \mid [G : H] = p^r$ for some $r \geq 1$. Hence, $|h| = p^e$ for some $e \geq 0$, and since G is nonabelian and simple, $|h| > 1$ (if not, $h \in \mathcal{Z}(G) \lhd G$, and $h \neq 1$, so by simplicity $\mathcal{Z}(G) = G$, a contradiction). Therefore, $|h| = p^e$ with $e \geq 1$. Hence, by Burnside's Thm 4, G cannot be simple, a contradiction.

As a corollary, suppose G has a abelian subgroup H of prime power index, then $H = \mathcal{Z}(H) = \{1\}$, therefore G is a p-group ($H \subsetneq G$ because G is nonabelian). But in this case $\mathcal{Z}(G) \neq \{1\}$, so by simplicity $\mathcal{Z}(G) = G$, contradicting the nonabelian condition for G. $\square$