Exercise 3: Prove the two formulas $\chi_S(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$ and $\chi_A(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$ stated in class.

Solution:

Since we have $T^2 = S \oplus A$, and $\chi_{T^2} = \chi^2$, we only need to prove one of these formulas: the linearity of characters will give $\chi^2 = \chi_S + \chi_A$. Let us choose χ_A.

Set $V = k\langle v_1, \ldots, v_n\rangle$ the corresponding kG-module associated to the character χ ($n = \dim_k V = \chi(1)$). We know that k-basis of T^2 is given by $v_i \otimes v_j$. Therefore, by simple computation we see that $A = k\langle v_i \otimes v_j - v_j \otimes v_i : i < j\rangle$ and this set of generators is a basis of S.

Set $g \in G$ and $D = D(g) \in \mathrm{Gl}_n(k)$. We have $g(v_i \otimes v_j) = (gv_i) \otimes (gv_j) = (\sum\limits_{k=1}^{n} D_{ki} v_k) \otimes (\sum\limits_{s=1}^{n} D_{sj} v_s) = \sum\limits_{k,s=1}^{n} D_{ki} D_{sj} (v_k \otimes v_s)$. Hence:

$$g \cdot (v_i \otimes v_j - v_j \otimes v_i) = \sum_{k,s=1}^{n} D_{ki} D_{sj} (v_k \otimes v_s) - \sum_{k,s=1}^{n} D_{kj} D_{si} (v_k \otimes v_s) =$$

$$= \sum_{k,s=1}^{n} (D_{ki} D_{sj} - D_{kj} D_{si})(v_k \otimes v_s) = \sum_{k<s} (D_{ki} D_{sj} - D_{kj} D_{si})(v_k \otimes v_s) +$$

$$+ \sum_{k>s} (D_{ki} D_{sj} - D_{kj} D_{si})(v_k \otimes v_s) = \sum_{k<s} (D_{ki} D_{sj} - D_{kj} D_{si})(v_k \otimes v_s) +$$

$$+ \sum_{k<s} (D_{si} D_{kj} - D_{sj} D_{ki})(v_s \otimes v_k) = \sum_{k<s} (D_{ki} D_{sj} - D_{kj} D_{si})(v_k \otimes v_s - v_s \otimes v_k)$$

Therefore, $\chi_S(g) = \sum\limits_{i<j} (D_{ii} D_{jj} - D_{ij} D_{ji})$.

On the other hand,

$$\chi(g)^2 = (\sum_i D_{ii})(\sum_j D_{jj}) = \sum_{i,j} D_{ii} D_{jj} = \sum_i D_{ii}^2 + 2 \sum_{i<j} D_{ii} D_{jj}.$$

To finish,

$$\chi(g^2) = tr(D(g^2)) = tr(D(g)^2) = \sum_i (D^2)_{ii} = \sum_i (\sum_j D_{ij} D_{ji}) = \sum_{i,j} D_{ij} D_{ji} = 2 \sum_{i<j} D_{ij} D_{ji} + \sum_i D_{ii}^2.$$

Therefore, $\chi(g)^2 - \chi(g^2) = 2 \sum\limits_{i<j} (D_{ii} D_{jj} - D_{ij} D_{ji}) = 2\chi_A(g)$ for all $g \in G$, and so

$$\chi_A(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)) \quad \forall\, g \in G\ .$$