Exercise 6: Let H be a subgroup of a (finite) abelian group G, and let U be a $\mathbb{C}G$-module with $\chi = \chi_U$. If $\chi(G \smallsetminus H) = 0$, show that $[G : H] \mid \dim U$.
(Hint: You may use the fact that any linear character $\lambda : H \to \mathbb{C}^*$ extends to a linear character $\lambda' : G \to \mathbb{C}^*$. Compute $[\chi, \lambda']$.)

Solution: Suppose we have a character τ on H that extends to a character τ' on G (we allow non-linear characters). Let us compute $[\chi, \tau']$:

$$[\chi, \tau'] = \frac{1}{|G|}(\sum_{h \in H} \chi(h)\tau'(h^{-1}) + 0) = \frac{|H|}{|G|}\frac{1}{|H|}\sum_{h \in H} \chi(h)\tau'(h^{-1}) = \frac{|H|}{|G|}[\chi_{|_H}, \tau'_{|_H}] = \frac{[\chi_{|_H}, \tau]}{[G : H]} .$$

Thus,

$$[G : H]\, [\chi, \tau'] = [\chi_{|_H}, \tau]. \qquad (*)$$

Note that if τ' is a virtual character, we have $[\chi, \tau'] \in \mathbb{Z}$ since $\chi = \chi_U$.

Now, let us choose a convenient pair (τ, τ'). Let τ be the regular character associated to $\mathbb{C}H$. We know that $\tau = \sum_{j=1}^{|H|} n_i \lambda_i$ for $n_i \geq 0$, where λ_i are the irreducible characters of H over $\mathbb{C}$, and hence linear characters (H is abelian). By our assumption, we can extend each λ_i to a linear character λ_i' over G which is afforded by λ' viewed as a representation. Hence, we can consider the *virtual* character $\tau' = \sum_{j=1}^{|H|} n_i \lambda_i'$.

In this way, we get $[\chi_{|_H}, \tau] = \frac{1}{|H|}\chi_{|_H}(1)|H| = \chi(1) = \dim U$, and since $[\chi, \tau'] \in \mathbb{Z}$, the result follows from (*). $\square$