

Exercise 2: For $\chi, \chi_1, \dots, \chi_m \in \text{Irr}(G)$ (any G), show that TFAE:

1. $\ker \chi_1 \cap \dots \cap \ker \chi_m \subseteq \ker \chi$.
2. χ appears in some product $\chi_1^{i_1} \cdots \chi_m^{i_m}$.
Show that (1), (2) are implied by, but not equivalent to:
3. $\chi \in \mathbb{Z}[\chi_1, \dots, \chi_m]$ (the subring of $\text{Ch}(G)$ gen. by $\{\chi_i\}$).

Solution:

• (2) \Rightarrow (1): Denote V, V_i the irreducibles $\mathbb{C}G$ -module afforded by the representations χ and χ_i resp. Since χ appears in some product $\chi_1^{i_1} \cdots \chi_m^{i_m}$, this means that $l = [\chi, \chi_1^{i_1} \cdots \chi_m^{i_m}] > 0$. W.l.o.g., we may assume $i_j > 0$ for all j . By construction of the $\mathbb{C}G$ -module structure of $(V_{1, \dots, m})^{\bar{i}}$, we have $\ker \chi_1^{i_1} \cdots \chi_m^{i_m} = \bigcap_{j=1}^m \ker(\chi_j^{i_j}) = \bigcap_{j=1}^m \ker \chi_j$ ($i_j > 0$ for all j).

Since V appear in the decomposition of $(V_{1, \dots, m})^{\bar{i}}$ with multiplicity $l > 0$, in particular, we can view V as a submodule of $(V_{1, \dots, m})^{\bar{i}}$. And so if $g \in \ker \chi_1 \cap \dots \cap \ker \chi_m = \ker \chi_1^{i_1} \cdots \chi_m^{i_m}$ this means that g acts trivially on $(V_{1, \dots, m})^{\bar{i}}$, thus, it acts trivially on V , so $g \in \ker \chi$ as we wanted to show.

If $i_j = 0$ for some j , then we get $\bigcap_{j=1}^m \ker \chi_j \subset \bigcap_{i_j \neq 0} \ker \chi_j \subset \chi$ and the result also holds.

• (1) \Rightarrow (2): We have $\ker \chi_1 \cap \dots \cap \ker \chi_m \subseteq \ker \chi$. As before, we know $\ker \chi_1 \cap \dots \cap \ker \chi_m = \ker(\chi_1 \cdots \chi_m) \subseteq \ker \chi$.

Set $N = \ker \chi_1 \cap \dots \cap \ker \chi_m \triangleleft G$. The condition $N \subseteq \ker \chi$, implies that V and $(V_{1, \dots, m})^{(1, \dots, 1)}$ are irreducible G/N -modules, afforded by the restricted characters $\chi'_1 \cdots \chi'_m$ and χ' . Moreover, in this case $\varphi' = \chi'_1 \cdots \chi'_m$ is a faithful character.

Hence, by Burnside-Brauer's Thm we have that the restricted character χ' is contained in some character $\{(\varphi')^j | j \in \mathbb{N}_0\}$, say $(\varphi')^{j_0}$. Thus, V is an irreducible component of $V_{1, \dots, m}^{j_0}$ as $\mathbb{C}(G/N)$ -modules. And since N acts trivial on both modules, then V is also an irreducible component of $V_{1, \dots, m}^{j_0}$ as $\mathbb{C}G$ -modules. Therefore, χ is contained in $\varphi^{j_0} = \chi_1^{j_0} \cdots \chi_m^{j_0}$. Hence, taking $i_1 = \dots = i_m = j_0$ the result follows.

• Due to the equivalence between (1) and (2), it suffices to show that (3) implies (2). Since $\chi \in [\chi_1, \dots, \chi_m]$, by definition, there exists $P \in \mathbb{Z}[X_1, \dots, X_m] \setminus \{0\}$, $P = \sum_{\substack{\alpha \in \mathbb{N}_0^m \\ \text{finite}}} b_\alpha X^\alpha$ s.t. $\chi =$

$P(\chi_1, \dots, \chi_m)$. Thus,

$$1 = [\chi, \chi] = [\chi, P(\chi_1, \dots, \chi_m)] = [\chi, \sum_{\alpha} b_\alpha \chi_1^{\alpha_1} \cdots \chi_m^{\alpha_m}] = \sum_{\alpha} b_\alpha [\chi, \chi_1^{\alpha_1} \cdots \chi_m^{\alpha_m}]$$

and $b_\alpha, [\chi, \chi_1^{\alpha_1} \cdots \chi_m^{\alpha_m}] \in \mathbb{Z}$ for all α . Therefore, for some α we have $b_\alpha [\chi, \chi_1^{\alpha_1} \cdots \chi_m^{\alpha_m}] \neq 0$, so in particular, $[\chi, \chi_1^{\alpha_1} \cdots \chi_m^{\alpha_m}] \neq 0$. Since χ is abs irreducible, by corollary of SOR this says that χ appears in the product $\chi_1^{\alpha_1} \cdots \chi_m^{\alpha_m}$.

To finish, we need to show that (1) \nRightarrow (3). For this, we will use Ex XV-1 b). Set $G = \mathbb{A}_4$, $\chi_1 = \varphi = (3, -1, 0, 0)$, $\chi = (1, 1, \omega, \omega^2)$. In this case, $[\varphi^2, \chi] = 1$ so χ appears in φ^2 . However, on that exercise we showed that $\chi \notin \mathbb{Z}[\varphi]$. \square