Exercise 1:

1. Let $\varphi = (2, 0, 1) \in \mathrm{Irr}(G)$ where $G = \mathbb{S}_3$. Show that the character ring $\mathrm{Ch}(G)$ (consisting of all virtual characters) is equal to $\mathbb{Z}[\varphi]$ (the ring generated by φ).

2. Let $\varphi = (3, -1, 0, 0) \in \mathrm{Irr}(G)$ where $G = \mathbb{A}_4$. Show that $\mathrm{Ch}(G)$ is not equal to $\mathbb{Z}[\varphi]$.

Solution:

1. In this case, we have the character table for $G = \mathbb{S}_3$:

	1	$(1\,2)$	$(1\,2\,3)$
	1	3	2
1_G	1	1	1
χ	1	-1	1
φ	2	0	-1

In this case, the character ring consists of the ring generated by all the differences of characters. Since product of characters correspond to the character of the tensor product of the modules they afford, we get that $\mathrm{Ch}(G)$ is the $\mathbb{Z}$-module generated by the irreducible characters $1_G, \chi, \varphi$. We what to show that $\mathrm{Ch}(G) = \mathbb{Z}[\varphi]$. Since we know $1_G, \varphi \in \mathbb{Z}[\varphi] \subseteq \mathrm{Ch}(G)$, we only need to show that $\chi \in \mathbb{Z}[\varphi]$. By Burnside-Brauer Theorem, since φ is a faithful character, χ should appear as a factor of φ^2, since it is no factor of 1_G nor φ.

In this case $\varphi^2 = (4, 0, 1)$, and so we only need to show that $[\varphi^2, \chi] = \pm 1$ (i.e., invertible over $\mathbb{Z}$). Let us compute this inner product. The conjugacy classes in G are $\{1\}, \{(1\,2), (1\,3), (2\,3)\}$ and $\{(1\,2\,3), (1\,3\,2)\}$. So:

$$[\varphi^2, \chi] = \frac{1}{6}(4 \cdot 1 + 0 + 2 \cdot 1 \cdot 1) = 1\ ,$$

which implies $\chi \in \mathbb{Z}[\varphi]$, as it remaind to show. □

2. In this case $G = \mathbb{A}_4$ and $\varphi = (3, -1, 0, 0) \in \mathrm{Irr}(G)$. Recall the character table for $\mathbb{A}_4$, were $G = \langle \tau; \sigma \rangle$, $\tau = (1\,2)(3\,4)$, $\sigma = (1\,2\,3)$:

	1	τ	σ	σ^2
	1	3	4	4
1_G	1	1	1	1
χ	1	1	ω	ω^2
μ	1	1	ω^2	ω
φ	3	-1	0	0

As before, we know that $\mathrm{Ch}(G) = \mathbb{Z}\langle 1_G, \chi, \mu, \varphi \rangle$. To show that $\mathrm{Ch}(G) \neq \mathbb{Z}[\varphi]$ we need to show that χ or $\mu \notin \mathbb{Z}[\varphi]$. By symmetry we have $[\varphi^n, \chi] = [\varphi^n, \mu]$ for all n. This will be the key-point.

Assume $\chi \in \mathbb{Z}[\varphi]$, then $\chi = \sum_{j=0}^{N} a_j \varphi^j$ for $a_j \in \mathbb{Z}$ and $N \in \mathbb{N}$. Since χ is abs irreducible, we get

$$1 = [\chi, \chi] = [\chi, \sum_{j=0}^{N} a_j \varphi^j] = \sum_{j=0}^{N} a_j [\chi, \varphi^j] = \sum_{j=0}^{N} a_j [\mu, \varphi^j] = [\mu, \sum_{j=0}^{N} a_j \varphi^j] = [\mu, \chi] = 0\ ,$$

a contradiction. Thus, $\mathrm{Ch}(G) \neq \mathbb{Z}[\varphi]$. □