**Exercise 1:** Let $k$ be a field of $\mathrm{char}(k) = 0$. Let $G = \langle h \rangle$ $(o(h) = 4)$ act on $V = kx \oplus ky$ by $h \cdot x = y$ and $h \cdot y = -x$. Use Molien's Formula to compute $P_{S(V)^G}$.

**Solution:**

By Molien's Formula we have:

$$P_{S(V)^G} = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - gt)} \ .$$

Thus, we need to compute $\det(I - gt) = t^2 \mathrm{char}_{D(g)}(t^{-1})$ where $V = kx \oplus ky$ and $\mathrm{char}_{D(g)}$ denotes the characteristic polynomial of the $k$-linear transformation given by multiplication by the matrix $D(g)$. In this case, the matrix corresponding to the linear transformations $D(g)$ are

$$D(1) = I_2 \quad ; \quad D(h) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \quad D(h^2) = D(h)^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad ;$$

$$D(h^3) = D(h^{-1}) = D(h)^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

Therefore, $\det(I - 1_G t) = t^2 \mathrm{char}_{D(1_G)}(t^{-1}) = t^2(t^{-1} - 1)^2 = (1-t)^2 = (t-1)^2$; $\det(I - ht) = t^2 \mathrm{char}_{D(h)}(t^{-1}) = t^2(t^{-2} + 1) = (t^2 + 1)$; $\det(I - h^2 t) = t^2 \mathrm{char}_{D(h^2)}(t^{-1}) = t^2(t^{-1} + 1)^2 = (t+1)^2$; $\det(I - h^3 t) = t^2 \mathrm{char}_{D(h^3)}(t^{-1}) = (t^2 + 1)$.

Hence:

$$P_{S(V)^G} = \frac{1}{4}\Big(\frac{1}{(1-t)^2} + \frac{2}{(1+t^2)} + \frac{1}{(1+t)^2}\Big) = \frac{1}{4} \frac{4 + 4t^4}{(1-t)^2(1+t)^2(1+t^2)} = \frac{1 + t^4}{(1-t)^2(1+t)^2(1+t^2)} =$$

$$= \frac{1+t^4}{(1-t^2)(1-t^4)}$$

as we wanted to show. □