Math 252 Exercises XIX #6

James Cook

2007 December 10

Exercise

Let $G = K \rtimes H$ be a Frobenius group in the usual notation.

(a) Show that $Z(G) = \{1\}$.

(b) Show that, for all $y \in G \setminus K$, the conjugation action of y on K defines an outer automorphism of K.

Solution

(a)

Let $g \in Z(G)$. For some $k \in K \setminus \{1\}$, $g \in C_G(k)$, so by Exercise 5, $g \in K$. For some $h \in H$, $h \in C_G(g)$, so again by Exercise 5, $g \notin K \setminus \{1\}$. So $g = 1$.

(b)

Let $y \in G \setminus K$. Conjugation by y is an automorphism of G, and since $K \triangleleft G$, it restricts to an automorphism on K. All that remains is to show that it is not an inner automorphism.

Suppose conjugation by y on K is equivalent to conjugation by some $k \in K$. Then $\forall \ell \in K\ \ell^y = \ell^k$, so $\ell^{(yk^{-1})} = \ell$. Picking any $\ell \in K \setminus \{1\}$, we have that $ky^{-1} \in C_G(\ell)$, so by Exercise 5, $ky^{-1} \in K$, so $y^{-1} \in K$. This is a contradiction: so conjugation by y is an outer automorphism on K.