# Math 252 Exercises XVII #3

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2007 November 19

## Exercise

Let $G = \langle \sigma, \varphi | \sigma^7 = \varphi^3 = 1, \sigma^\varphi = \sigma^2 \rangle$ and $\zeta = e^{2\pi i/7}$. In Chapter 0, we have come up with the representations $D, D' : G \to \mathrm{GL}_3(\mathbb{C})$ where $D(\sigma) = \mathrm{diag}(\zeta, \zeta^2, \zeta^4)$, $D'(\sigma) = \mathrm{diag}(\zeta^6, \zeta^5, \zeta^3)$, and $D(\varphi) = D'(\varphi) = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. Show that both $D$ and $D'$ are monomial representations.

## Solution

Let $H = \langle \sigma \rangle \subseteq G$ and let $\tau \in \mathbb{C}$ have order 7. Let $V = \mathbb{C}x$ be a $\mathbb{C}H$-module with $H$-action $\sigma \cdot x = \tau x$.

Note that $H$, $\varphi H$ and $\varphi^2 H$ are the cosets of $H$ in $G$, so $\{1 \otimes x, \varphi \otimes x, \varphi^2 \otimes x\}$ is a basis for $V^G$. We have

$$\begin{aligned}
\sigma \cdot (1 \otimes x) =& \sigma \otimes x \\
=& 1 \otimes \sigma x \\
=& \tau (1 \otimes x) \\
\sigma \cdot (\varphi \otimes x) =& \sigma\varphi \otimes x \\
=& \varphi \otimes \sigma^2 x \\
=& \tau^2 (\varphi \otimes x) \\
\sigma \cdot (\varphi^2 \otimes x) =& \sigma\varphi^2 \otimes x \\
=& \varphi^2 \sigma^4 \otimes x \\
=& \tau^4 (\varphi^2 \otimes x).
\end{aligned}$$

The action of $\sigma$ on $V^G$ is represented by the matrix $\mathrm{diag}(\tau, \tau^2, \tau^4)$. It is easily seen that the action of $\varphi$ is represented by the matrix $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. Taking $\tau = \zeta$, $V^G$ is the $\mathbb{C}G$-module corresponding to $D$, and taking $\tau = \zeta^6$, $V^G$ is the $\mathbb{C}G$-module corresponding to $D'$.