Math 252 Exercises XVII #1

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2007 November 19

Exercise

Let $k = \mathbb{Q}$ and $G = A_4$. Let χ by any one of the three nontrivial linear characters on the normal 2-Sylow group of G. Show that χ^G is the tetrahedral character on G (which is also the character of the 3-dimensional reduced permutation representation of G on the G-set $\{1, 2, 3, 4\}$).

Solution

Let λ be the tetrahedral character on G.

Let $H = \{1, u, v, w\}$ be the 2-Sylow subgroup of G isomorphic to the Klein 4-group. Fix any one-dimensional $\mathbb{Q}H$-module V with nontrivial character χ: then $\chi(h) = 1$ for one $h \in \{u, v, w\}$ and $\chi(h) = -1$ for the other two $h \in \{u, v, w\}$. Without loss of generality, let $\chi(u) = 1$ and $\chi(v) = \chi(w) = -1$.

Let $r \in G$ have order 3. Note that $\{H, rH, r^2H\}$ are all cosets of H in G, and conjugation by r is an order-3 permutation on $\{u, v, w\}$. Without loss of generality, assume $u^r = v$, $v^r = w$ and $w^r = u$. (If this is not the case, replace r with r^{-1} and it will become true.)

Choose $0 \neq x \in V$. Then $\{1 \otimes x, r \otimes x, r^2 \otimes x\}$ forms a basis for V^G.

$$\begin{aligned}
u \cdot (1 \otimes x) =& u \otimes x \\
=& 1 \otimes u \cdot x \\
=& 1 \otimes x \\
u \cdot (r \otimes x) =& ur \otimes x \\
=& ru^r \otimes x \\
=& rv \otimes x \\
=& r \otimes v \cdot x \\
=& -(r \otimes x) \\
u \cdot (r^2 \otimes x) =& ur^2 \otimes x \\
=& w^r r^2 \otimes x \\
=& r^2 w \otimes x \\
=& r^2 \otimes w \cdot x \\
=& -(r^2 \otimes x).
\end{aligned}$$

From the above it follows that

$$\chi^G(u) = -1 = \lambda(u).$$

Now,

$$\begin{aligned}
r \cdot (1 \otimes x) =& r \otimes x \\
r \cdot (r \otimes x) =& r^2 \otimes x \\
r \cdot (r^2 \otimes x) =& 1 \otimes x.
\end{aligned}$$

Above we see that $\chi^G(r) = 0 = \lambda(r)$, and by a similar argument, $\chi^G(r^2) = 0 = \lambda(r^2)$.

Clearly $\chi^G(1) = \lambda(1)$. 1, u, r and r^2 represent all conjugacy classes of G, so we have determined that $\chi^G = \lambda$.