Math 252 Exercises XV #6

James Cook

2007 November 9

Exercise

Let H be a subgroup of a finite abelian group G, and let U be a $\mathbb{C}G$-module with $\chi = \chi_U$. If $\chi(G \setminus H) = 0$, show that $[G : H] \mid \dim U$. (*Hint.* You may use the fact that any linear character $\lambda : H \to \mathbb{C}^*$ extends to a linear character $\lambda' : G \to \mathbb{C}^*$. Compute $[\chi, \lambda']$.)

Solution

Let $n = |G|$ and $m = |H|$. Consider $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$ as groups under pointwise multiplication. Then the restriction operation $r : \mathrm{Irr}(G) \to \mathrm{Irr}(H)$ via $\lambda \mapsto \lambda|_H$ is a group homomorphism. Let $K = \ker r$. $|K| = n/m$.

Now, assume $\chi_U(G \setminus H) = 0$, and take any $\lambda_0, \lambda_1 \in \mathrm{Irr}(G)$ such that $r(\lambda_0) = r(\lambda_1)$. Then

$$
\begin{aligned}
&[\chi_U, \lambda_0] \\
=&\sum_{g\in G} \chi_U(g)\lambda_0(g^{-1}) \\
=&\sum_{h\in H} \chi_U(g)\lambda_0(g^{-1}) \\
=&\sum_{h\in H} \chi_U(g)\lambda_1(g^{-1}) \\
=&\sum_{g\in G} \chi_U(g)\lambda_1(g^{-1}) \\
=&[\chi_U, \lambda_1].
\end{aligned}
$$

Thus, noting that G is the disjoint union of $r^{-1}(\{\lambda'\})$ for $\lambda' \in \mathrm{Irr}(H)$,

$$\begin{aligned}
&\chi_U(1) \\
&= \left(\sum_{\lambda \in \mathrm{Irr}(G)} [\chi_U, \lambda] \lambda \right)(1) \\
&= \sum_{\lambda \in \mathrm{Irr}(G)} [\chi_U, \lambda] \lambda(1) \\
&= \sum_{\lambda \in \mathrm{Irr}(G)} [\chi_U, \lambda] \\
&= \sum_{\lambda' \in \mathrm{Irr}(H)} \sum_{\lambda \in r^{-1}(\lambda')} [\chi_U, \lambda]
\end{aligned}$$

(noting that $[\chi_U, \lambda]$ does not depend on the choice of representative $\lambda \in r^{-1}(\lambda')$, and fixing some λ for each λ')

$$= \sum_{\lambda' \in \mathrm{Irr}(H)} \frac{n}{m} [\chi_U, \lambda].$$

In particular, $n/m = [G : H] \mid \dim U = \chi_U(1)$.