Math 252 – Exercises XIV #1

James Cook – james.cook@utoronto.ca

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Problem

Let $G = \langle h | h^4 = 1 \rangle$ act on $V = kx \oplus ky$ by $hx = y$ and $hy = -x$. Use Molien's Formula to compute $P_{S(V)^G}$.

Solution

A h-action can be represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Using this,

$$\det(I - t) = \det \begin{pmatrix} 1-t & 0 \\ 0 & 1-t \end{pmatrix} = (1-t)^2$$

$$\det(I - ht) = \det \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} = 1 + t^2$$

$$\det(I - h^2 t) = \det \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix} = (1+t)^2$$

$$\det(I - h^3 t) = \det \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} = 1 + t^2.$$

By Molien's formula,

$$\begin{aligned} P_{S(V)^G}(t) &= \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - gt)} \\ &= \frac{1}{4} \sum_{i=0}^{3} \frac{1}{\det(I - h^i t)} \\ &= \frac{1}{4} \left[\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2} + \frac{2}{1+t^2} \right] \end{aligned}$$

using a common denominator:

$$
\begin{aligned}
&= \frac{(1+t)^2(1+t^2)+(1-t)^2(1+t^2)+2(1-t)^2(1+t)^2}{4(1-t)^2(1+t)^2(1+t^2)} \\
&= \frac{[(1+t)^2+(1-t)^2](1+t^2)+2(1-t^2)^2}{4(1-t)^2(1+t)^2(1+t^2)} \\
&= \frac{2(1-t^2)^2+2(1-t^2)^2}{4(1-t)^2(1+t)^2(1+t^2)} \\
&= \frac{4(t^4+1)}{4(1-t)^2(1+t)^2(1+t^2)} \\
&= \frac{t^4+1}{(1-t)^2(1+t)^2(1+t^2)}
\end{aligned}
$$

Noting that $(1-t)^2(1+t)^2(1+t^2) = (1-t^2)^2(1+t^2) = (1-t^2)(1-t^4)$, we arrive at the form shown in class:

$$
= \frac{t^4+1}{(1-t^2)(1-t^4)}.
$$