MAT 252 - HW #20

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1. **Fun Problem.** Suppose you find the following partial character table for a group G of order 72 where the conjugacy class sizes have been indicated. Can you complete it?

	g_1	g_2	g_3	g_4	g_5	g_6
	1	8	9	18	18	18
χ_2	1	1	1	-1	-1	1
χ_3	1	1	1	-1	1	-1

Proof: Firstly, by tensor product, we get a new 1-dimensional character $\chi_4 = \chi_2\chi_3 = (1, 1, 1, 1, -1, -1)$ which is surely irreducible. Let χ_1 be the trivial character and the rest of the characters be χ_5, χ_6. By Magic equation, we have

$$|G| = 72 = 1^2 + 1^2 + 1^2 + 1^2 + \chi_5(1)^2 + \chi_6(1)^2 \Longrightarrow 68 = \chi_5(1)^2 + \chi_6(1)^2.$$

Running through the possible integers, we find that $\chi_5(1), \chi_6(1)$ must be 2 or 8, say in this order. So, the character table becomes

	g_1	g_2	g_3	g_4	g_5	g_6
	1	8	9	18	18	18
χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	1
χ_3	1	1	1	-1	1	-1
χ_4	1	1	1	1	-1	-1
χ_5	2	a_2	a_3	a_4	a_5	a_6
χ_6	8	b_2	b_3	b_4	b_5	b_6

where the a_i's, b_i's are real because there are only one character of dimension 2 and one of 8.

By FOR, since $[\chi_5, \chi_i] = 0$ for $i = 1, 2, 3, 4$, we have

$$\begin{aligned} 0 &= 2 + 8a_2 + 9a_3 + 18a_4 - 18a_5 - 18a_6 \\ 0 &= 2 + 8a_2 + 9a_3 - 18a_4 + 18a_5 - 18a_6 \\ 0 &= 2 + 8a_2 + 9a_3 - 18a_4 - 18a_5 + 18a_6 \\ 0 &= 2 + 8a_2 + 9a_3 + 18a_4 - 18a_5 + 18a_6 \end{aligned}$$

Note that we can solve for a_3, a_4, a_5, a_6 in terms of a_2 and so we get $a_3 = -(2 + 8a_2)/9$ and $a_4 = a_5 = a_6 = 0$. Then, again by FOR that $[\chi_5, \chi_5] = 1$, we have $72 = 2^2 + 8a_2^2 + 9(2 + 8a_2)^2/81$. We solve for a_2 and found it to be either 2 or $-38/17$. Since the sum of a row of a character table is a non-negative integer by Ex.11-5, $a_2 = 2$ and $a_3 = -2$.

By a similar argument, we found $b_2 = -1$, $b_3 = 0$ and $b_4 = b_5 = b_6 = 0$.