MAT 252 - HW #16

by Ka Choi

3. Prove the two formulas $\chi_S(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$ and $\chi_A(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$ stated in class.

   Proof: Recall that, in class, we have seen that $S \cong S^2(V)$ and $A \cong \Lambda^2(V)$. It suffices to prove one of the above statements since $S^2(V) \oplus \Lambda^2(V) = T^2(V)$ and so $\chi_S + \chi_A = \chi^2$. Now, we fix a $g \in G$. If $\{v_i\}$ is an eigenbasis of V with respect to g with eigenvalues $\{\lambda_i\}$, then $\{v_i \wedge v_j\}_{i<j}$ is a basis of $\Lambda^2(V)$. Since $g \cdot (v_i \wedge v_j) = g \cdot v_i \wedge g \cdot v_j = \lambda_i \lambda_j (v_i \wedge v_j)$, we see that

   $$\chi_A(g) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}((\sum_i \lambda_i)^2 - \sum_i \lambda_i^2) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$$

   as $g^2 \cdot v_i = g \cdot (g \cdot v_i) = g \cdot \lambda_i v_i = \lambda_i^2 v_i$.