GROUPS WHOSE IRREDUCIBLE REPRESENTATIONS HAVE DIMENSION ≤ 2

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Problem 1. *Let G be a group all of whose irreducible representations have dimension less than or equal to 2. Write the order of G as $n = 2^b a$ where a is odd. Then G has a unique normal Abelian subgroup A of order a and G can be written as a semidirect product of A and a 2-group.*

Proof. The proof is by induction on b. If $b = 0$, then take $A = G$, which must be Abelian since $|G|$ odd implies that there are no 2-dimensional characters. Now suppose $b > 0$. Let $d > 0$ be the number of 2-dimensional irreducible representations an ℓ be the number of linear characters. Then the magic equation tells us that $\ell + 4d = n = 2^b a$, so ℓ is even. Let N' be the subgroup of G^{ab} of all elements of odd order, which is proper since $\ell = |G|$ is even. Let $N \lhd G$ be the preimage of N'. Since the index $[G^{\mathrm{ab}} : N']$ is a power of 2, N has order $2^{b'} a$ where $b' < b$. Every irreducible representation of N is a subrepresentation of the restriction of a representation from G, so all irreducible representations of N have dimension at most 2. Thus, we can apply the inductive hypothesis to show that N has a unique normal Abelian subgroup A of order a. Conjugation by an element of G induces an automorphism of N. Since A is unique, it is fixed by this conjugation, so A is a normal subgroup of G. Finally, to see that G is a semidirect product, note that the 2-Sylow subgroup must intersect A trivially. $\square$