# MATH 252 EXERCISES XVI PROBLEM 7

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## 1. Problem:

Let $\chi \in \text{Irr}(G)$ over a group $G$. If $\chi$ comes from a real representation, show that $s(\chi) = 1$.

## 2. Solution:

Suppose $\chi$ is the character of a real representation on an $\mathbb{R}G$-module $V$. Let $v_1, \cdots, v_n$ be a basis of $V$. Consider $\langle v_i, v_j \rangle$, the standard dot product using our chosen basis; it is a positive definite symmetric bilinear form. However, we would like it to be $G$-invariant so as to induce a $\mathbb{R}G$-morphism on $\text{Sym}^2(V)$. To achieve this, rescale by defining a new inner product $\langle v_i, v_j \rangle_G := \sum_{g \in G} \langle gv_i, gv_j \rangle$; this is clearly $G$-invariant, and retains the desired properties. Now this induces a nonzero linear $\mathbb{R}G$-morphism on $\text{Sym}^2(V)$ defined by $\lambda : \text{Sym}^2(V) \to \mathbb{R}$, $\lambda(v_i \otimes v_j) = \langle v_i, v_j \rangle_G$. Since this is a nonzero morphism to the trivial $\mathbb{R}G$-module, we have $[\chi_S, 1_G] \geq 1$. But now observe that by a property of the Frobenius-Schur indicator as described in a corollary in class, $[\chi_S, 1_G] \leq 1$ with equality iff $[\chi_A, 1_G] = 0$ and $s(\chi) = 1$, as desired (since $[\chi_S, 1_G] + [\chi_A, 1_G] = [\chi^2, 1_G] = [\chi, \bar{\chi}] = 0$ or 1 depending on whether $\chi$ is real, so each of the nonnegative summands on the left is at most 1, and $s(\chi) = [\chi_S, 1_G] - [\chi_A, 1_G] = 1 - 0 = 1$).

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