MATH 252 EXERCISES XVI PROBLEM 5

DAVID BERLEKAMP

1. Problem:

Let G be a finite nonabelian simple group. Show that any subgroup $H \subset G$ of prime-power index is centerless (that is, $Z(H) = \{1\}$). Deduce from this that G has no abelian subgroup of prime-power index.

2. Solution:

Suppose $g \in Z(H)$. Since g commutes with every element of H, its centralizer in G contains H; so the conjugacy class of g in G has size dividing $[G : H]$, i.e. is a prime power by supposition on H. By Burnside's Theorem 4, G is then not simple unless the conjugacy class of g has size 1. However, then $g \in Z(G) = \{1\}$. Thus any subgroup of prime-power index cannot be abelian (unless it is trivial, in which case G itself is a p-group and solvable).

Date: Nov. 2007.