MATH 252 EXERCISES XV PROBLEM 4

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1. Problem:

Prove the following generalization of Frobenius' Integrality Theorem: For $\chi \in \mathrm{Irr}(G)$, $g \in G$, and $d = \gcd(\chi(1), |g|)$, $d\chi(g)/\chi(1)$ is an algebraic integer.

2. Solution:

Let $r := |g|/\chi(1)$. By the original form of Frobenius' Integrality Theorem, $r\chi(g)$ is an algebraic integer. As this is a rational multiple of $\chi(g)$, they have the same degree m over $\mathbb{Q}$. The latter is also an algebraic integer, so let their respective (monic) minimal polynomials be $p, q \in \mathbb{Z}[x]$, and write $q(x) = \sum_{i=0}^{m} a_i x^i$.

Now note that $f := \sum_{i=0}^{m} r^{m-i} a_i x^i \in \mathbb{Q}[x]$ satisfies $f(r\chi(g)) = r^m q(\chi(g)) = 0$, so $p \mid f$. But they are monic of the same degree, so in fact $p = f$, and $f \in \mathbb{Z}[x]$. Thus $\chi(1)^i \mid |g|^i a_{m-i}$ for each $i \in [m]$. Writing $|g| = de$, we see that for each $i \in [m]$, $\chi(1)^i \mid d^i a_{m-i}$ (because $\chi(1)^i \mid (de)^i a_{m-i}$ and $\chi(1)$ is relatively prime to e). Then the monic polynomial $h := \sum_{i=0}^{m} (d/\chi(1))^{m-i} a_i x^i \in \mathbb{Z}[x]$ satisfies $h(d\chi(g)/\chi(1)) = (d/\chi(1))^m q(\chi(g)) = 0$, i.e. $d\chi(g)/\chi(1)$ is an algebraic integer.

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