MATH 252 EXERCISES XV PROBLEM 3

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1. Problem:

Let U, V, W be simple kG-modules, where k is a splitting field of char0 for G. Let T be the regular module kG.

(a) Show that $U \otimes T \equiv (\dim U)T$.

(b) Show that $U \otimes V$ can be embedded in T.

(c) Show that W appears in $U \otimes V$ at most $\dim W$ times.

2. Solution:

Throughout the following we write $d_Y := \dim Y$ for Y a kG-module.

(a) We have $\chi_T(g) = \delta_{1g}$, so $\chi_{U\otimes T}(g) = \chi_U(g)\chi_T(g) = \chi_U(g)\delta_{1g} = \chi_U(1)\delta_{1g} = d_U\delta_{1g} = d_U\chi_T(g) = \chi_{d_U T}(g)$. Since k is a splitting field of char0 identity of these characters is sufficient to determine isomorphism of the representations, so $U \otimes T \simeq d_U T$.

(c) The number of times m that W appears in $U \otimes V$ is at most $\langle \chi_{U\otimes V}, \chi_W \rangle = \langle \chi_U \chi_V, \chi_W \rangle = \langle \chi_U, \bar{\chi}_V \chi_W \rangle$, the number of times U appears in $V^* \otimes W$. This is at most $\frac{d_V d_W}{d_U}$. Similarly, m is the number of times V appears in $W \otimes U^*$, which is at most $\frac{d_U d_W}{d_V}$. Then multiplying gives $m^2 \leq d_W^2 \Rightarrow m \leq d_W$, as desired.

(b) The number of times that any simple W appears in T is d_W; therefore by (c) there are enough factors of each W to embed $U \otimes V$ in T.

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