MATH 252 EXERCISES XV PROBLEM 2

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1. Problem:

For $\chi, \chi_1, \cdots, \chi_m \in \mathrm{Irr}(G)$ (any G), show that TFAE:
(a) $\ker \chi_1 \cap \cdots \cap \ker \chi_m \subset \ker \chi$.
(b) χ appears in some product $\chi_1^{i_1} \cdots \chi_m^{i_m}$.
Show that (a), (b) are implied by, but not equivalent to:
(c) $\chi \in \mathbb{Z}[\chi_1, \cdots, \chi_m]$ (the subring of $\mathrm{Ch}(G)$ generated by $\{\chi_i\}$).

2. Solution:

Note that the intersection of kernels is the kernel of the product (since the identity map on the tensor product of representations is the tensor product of the identity maps).

(a) $\Rightarrow$ (b): $H := \bigcap_j \ker \chi_j \triangleleft G$. As H is contained in the kernel of every character under consideration, they correspond to characters $\{\bar{\chi}_j\}, \bar{\chi}$ of G/H. Now $\{1\} = \bigcap_j \ker \bar{\chi}_j = \ker \prod_j \bar{\chi}_j = \ker \bar{\tau}$, where $\tau := \prod_j \chi_j$. Thus $\bar{\tau}$ is the character of a faithful representation of G/H, so by Burnside-Brauer, $\bar{\chi}$ appears in some $\bar{\tau}^n$; but then χ appears in $\tau^n = \chi_1^n \cdots \chi_m^n$.

(b) $\Rightarrow$ (a): Each $g \in \ker \chi_1^{i_1} \cdots \chi_m^{i_m}$ acts as the identity on the corresponding representation; in particular, on the subspace corresponding to the summand χ, so $g \in \ker \chi$. Also, we always have $\ker \chi_j \subset \ker \chi_j^n$ for any n, j. Then we have $\bigcap_j \ker \chi_j \subset \bigcap_j \ker \chi_j^{i_j} = \ker \chi_1^{i_1} \cdots \chi_m^{i_m} \subset \ker \chi$.

(c) $\Rightarrow$ (b): The products $\chi_1^{i_1} \cdots \chi_m^{i_m}$ generate $\mathbb{Z}[\chi_1, \cdots, \chi_m]$ as a $\mathbb{Z}$-module. If χ were orthogonal to all of these, χ would be orthogonal to itself, contradicting $\chi \in \mathrm{Irr}(G)$.

(b) $\not\Rightarrow$ (c): If χ_1 is a faithful real irreducible character of any group with non-real irreducible character χ, then χ will appear in some power of χ_1 (satisfying (b)) but $\chi \notin \mathbb{Z}[\chi_1]$, as in problem 1 (e.g., take $G = A_4$, $\chi_1 = (3, -1, 0, 0)$, $\chi = (1, 1, \omega, \omega^2)$).

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