MATH 252 EXERCISES XV PROBLEM 1

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1. Problem:

(a) Let $\phi = (2, 0, 1) \in \mathrm{Irr}(G)$ where $G = S_3$. Show that the character ring $\mathrm{Ch}(G)$ (consisting of all virtual characters) is equal to $\mathbb{Z}[\phi]$ (the ring generated by ϕ).

(b) Let $\phi = (3, -1, 0, 0) \in \mathrm{Irr}(G)$ where $G = A_4$. Show that $\mathrm{Ch}(G)$ is *not* equal to $\mathbb{Z}[\phi]$.

2. Solution:

(a) $\mathrm{Ch}(G)$ is generated by $\mathrm{Irr}(G) = \{1_G, \psi, \phi\}$, where $\psi = (1, -1, -1)$. The inclusion $\mathbb{Z}[\phi] \subset \mathrm{Ch}(G)$ is thus trivial. As for the other, we note that $1_G = \phi^0 \in \mathbb{Z}[\phi]$. Further, $\phi^2 = (4, 0, 1)$ satisfies $\langle \phi^2, \chi \rangle = 1$ for $\chi \in \mathrm{Irr}(G)$, so $\psi = \phi^2 - \phi - 1_G$. Thus $\mathrm{Irr}(G) \subset \mathbb{Z}[\phi]$, so $\mathrm{Ch}(G) \subset \mathbb{Z}[\phi]$ as desired.

(b) Note that A_4 has non-real characters, such as $(1, 1, \omega, \omega^2)$ (where $\omega = e^{2\pi i/3}$). But $\phi = (3, -1, 0, 0)$ is real, so each element of $\mathbb{Z}[\phi]$ is also real. Thus $\mathrm{Ch}(G) \supsetneq \mathbb{Z}[\phi]$.

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