

MAT252 Lecture

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This is a course in finite representation theory. In general, all representations we deal with will be finite-dimensional, and all groups finite.

1 The classical definition

Classically, an n -dimensional representation of G is a group homomorphism $D : G \rightarrow GL_n(k)$, where k is (and will be for the duration of the page) a commutative ring. D is from the German *Darstellung*.

Two representations of dimension n are *equivalent* (written $D \sim D'$) if $\exists T \in GL_n(k) : \forall g : T^{-1}D(g)T = D'(g)$.

The *character* of D is a function $\chi_D : G \rightarrow k$ given by $\chi_D(g) = \text{tr } D(g)$. If $D \sim D'$ then $\chi_D = \chi_{D'}$. Also, if $g \sim g'$ (\sim being conjugacy in G) then $\chi_D(g) = \chi_D(g')$. Thus we call χ_D a *class function*: it is constant on the conjugacy classes.

We say D is *faithful* iff it's injective, *trivial* iff it's the 0 map.

2 The modern definition

From the modern perspective, a G -module over k is a k -module (for us, usually finitely generated) with a G -action $(g, v) \mapsto g \cdot v$ such that

1. $g'(gv) = (g'g)v$
2. $1 \cdot v = v$
3. $v \mapsto gv$ is k -linear, so that $g \in G$ acts as a k -automorphism (g^{-1} exists).

Example. \mathbb{F}_8 as an \mathbb{F}_2 -module, with S_8 acting by permuting \mathbb{F}_8 , violates (3).

Isomorphism of G -modules over k is defined in the obvious fashion. G acts *faithfully* iff $(\forall v : gv = v) \Rightarrow g = 1$.

To reconcile these definitions: if $D : G \rightarrow GL_n(k)$, we can see k^n as a G -module over k by $gv = D(g)v$. Conversely, if $V = k^n$ is a free G -module over k , fixing a base on V gives us matrices relative to it.

$D, D' : G \rightarrow GL_n(k)$ are equivalent iff the two corresponding G -modules are equivalent.

*amended and based on an original by Alex Fink

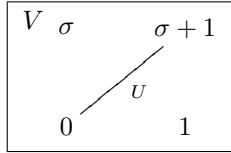
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Now restrict k to be a field. We form the group ring $kG = k[G] = \{\sum_{\text{finite}} a_g g : a_g \in k, g \in G\}$ of formal sums, writing G multiplicatively. kG is a k -algebra.

Observe that G -modules over k are the same as kG -modules.

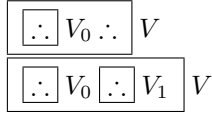
The representation afforded by a G -module $V \neq 0$ over k is *irreducible*, respectively *indecomposable*, iff \nexists a nontrivial kG -submodule of V , respectively $\nexists V_1, V_2 \neq 0 : V = V_1 \oplus V_2$ (V_1 and V_2 non-trivial submodules). Irreducible is strictly stronger than indecomposable. 1-dimensional representations are always irreducible.

Example. Let $k = \mathbb{F}_2$, $G = \langle \sigma \mid \sigma^2 = 1 \rangle$, and $V = {}_{kG}(kG)$, i.e. kG as a kG -module by left-multiplication. This is called the left regular kG -module. $V = \{0, 1, \sigma, \sigma + 1\}$. V is not irreducible, since it has a submodule $U = \{0, \sigma + 1\}$ with trivial G -action. But U is the only such nontrivial submodule, by exhaustion, so V is indecomposable. (The key here: $\text{char } k \mid \#G$.)



Matricially, D is reducible iff $\exists T \in GL_n(k) : T^{-1}D(g)T = \begin{pmatrix} D_1(g) & E(g) \\ 0 & D_2(g) \end{pmatrix} \forall g$

since if $V \subseteq V_0$ is a submodule and we extend a basis of V_0 to one of V , then V_0 affords $g \mapsto D_1(g)$, and V/V_0 affords $g \mapsto D_2(g)$. D is decomposable iff $\exists T : T^{-1}D(g)T = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \forall g$, since $V = V_0 \oplus V_1 \dots$



Note that one-dimensional representations are homomorphisms $G \rightarrow k^*$, and for these representations $D \sim D'$ iff $D \equiv D'$.

4 Composition factors

Let V be a kG -module. Then \exists a *composition series* $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_m = V$ such that V_{i+1}/V_i are simple kG -modules. By Jordan-Hölder, the multiset of factors V_{i+1}/V_i is uniquely determined.

Example. In our last example, the decomposition factors are $\{k, k\}$ with trivial G -action.

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The direct sum $V \oplus V'$ of representations V and V' has matrix form $g \mapsto \begin{pmatrix} D(g) & 0 \\ 0 & D'(g) \end{pmatrix}$.

$\chi_{V \oplus V'} = \chi_V + \chi_{V'}$.

Scalar extension is trivial from the matricial viewpoint. $G \rightarrow GL_n(k) \subseteq GL_n(K)$ (K a field extension of k). From the module optic, let V be a kG -module K/k . We form the scalar extension $K \otimes_k V =: V^k$ with K -action $g(\alpha \otimes v) = \alpha \otimes gv$.