Exercise XVI.8. *Using the character table for the simple group $G = PSL_2(\mathbb{F}_7)$, compute the Frobenius-Schur indicator for all $\chi \in Irr(G)$.*

[Anne Shiu] We will show that the irreducible characters χ_i (ordered in non-decreasing order of dimension), have the following indicators:

$$s(\chi_1) = s(\chi_4) = s(\chi_5) = s(\chi_6) = 1, \qquad \text{and} \qquad s(\chi_2) = s(\chi_3) = 0.$$

From the literature we know that the unique conjugacy class of size 21 is the set of involutions. Therefore, from the involution count formula and the character table's first column, we have

$$\begin{aligned} 21 &= \sum_{\chi \neq 1} s(\chi)\chi(1) \\ &= 3s(\chi_2) + 3s(\chi_3) + 6s(\chi_4) + 7s(\chi_5) + 8s(\chi_6) \\ &= 0 + 0 + 6s(\chi_4) + 7s(\chi_5) + 8s(\chi_6), \end{aligned}$$

where the last equality holds because the two three-dimensional characters are not real. Finally, the remaining coefficients have sum 6+7+8=21, while the $s(\chi_4)$, $s(\chi_5)$, $s(\chi_6)$ are ± 1; therefore they must be 1.

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