Exercise XV.7. *Does the conclusion of Exercise 6 hold if H is abelian, but G is not?*

[Anne Shiu] No, Exercise 6 does not hold if G is non-abelian. To show this, we must find a non-abelian group G with an abelian subgroup H, such that G admits a character χ with $\chi(G \setminus H) = 0$ and $[G\colon H] \nmid \chi(1)$. To this end, let G be the quaternion group of order 8, and let H be the group generated by -1. We see that H is abelian. Now let χ be the unique two-dimensional irreducible character of G

$$\chi = (2, -2, 0, 0, 0),$$

where the conjugacy class representatives are $(1, -1, i, j, k)$. We see then that

$$\chi(G \setminus H) = 0, \qquad \text{and} \qquad [G\colon H] = 4 \nmid 2 = \chi(1).$$

Hence our G, H, and χ display the desired properties.

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