Exercise XV.1.
(a) Let $\phi = (2, 0, 1) \in Irr(G)$ where $G = S_3$. Show that the character ring $Ch(G)$ (consisting of all virtual characters) is equal to $\mathbb{Z}[\phi]$ (the ring generated by ϕ).
(b) Let $\phi = (3, -1, 0, 0) \in Irr(G)$ where $G = A_4$. Show that $Ch(G)$ is not *equal to $\mathbb{Z}[\phi]$.*

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(a)
The character ring of G is

$$Ch(G) = \{\chi_V - \chi_{V'} \mid V, V' \ kG-\text{modules}\} \tag{1}$$
$$= \{b_1\chi_1 + b_2\chi_2 + b_3\chi_3 \mid b_i \in \mathbb{Z}\}. \tag{2}$$

The second ring is

$$\mathbb{Z}[\phi] = \left\{ \sum_{i=0}^{r} a_i \phi^i \mid r \geq 0, \ a_i \in \mathbb{Z} \right\}, \tag{3}$$

where we view constants a_0 as constant functions on G.

The inclusion $\mathbb{Z}[\phi] \subset Ch(G)$ is clear because any power ϕ^i is a $\mathbb{Z}$-linear combination of the irreducible characters χ_i. To verify the reverse inclusion, we need only show that each of the three irreducible characters χ_i is in $\mathbb{Z}[\phi]$. Now, $\chi_1 = (1, 1, 1)$ is simply 1 in the ring $\mathbb{Z}[\phi]$. Next, $\chi_2 = (1, -1, 1) = 1 + \phi$. Finally, $\chi_3 = \phi$, which completes our proof.

(b)
We verify that $\mathbb{Z}[\phi] \not\subset Ch(G)$ by showing that $\chi_2 = (1, 1, \omega, \omega^2) \notin \mathbb{Z}[\phi]$. Now any $\phi(123) = 0$, so any $f \in \mathbb{Z}[\phi]$ satisfies $f(123) \in \mathbb{Z}$. However $\chi_2(123) = \omega$, which is not an integer; hence $\chi_2 \notin \mathbb{Z}[\phi]$, as claimed.

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